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Li–Yau–Hamilton estimates and Bakry–Emery–Ricci curvature

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ABSTRACT

In this paper we derive Cheng–Yau, Li–Yau, Hamilton estimates for Riemannian manifolds with Bakry–Emery–Ricci curvature bounded from below, and also global and local upper bounds, in terms of Bakry–Emery–Ricci curvature, for the Hessian of positive and bounded solutions of the weighted heat equation on a closed Riemannian manifold.

1. Introduction

In a seminal paper [27], Li and Yau derived the gradient estimate and Harnack inequality for positive solutions of heat equation on a complete Riemannian manifold. Li–Yau estimate has been improved and generalized to other nonlinear equations on a Riemannian manifold, see [1,2,5,7,9,14,16,19–24,29–31,34–36,39,40,42] and references therein.

An important generalization is a diffusion operator

$$\Delta_V := \Delta + \langle V, \nabla \rangle \tag{1.1}$$

on a Riemannian manifold $(\mathcal{M}, g)$ of dimension m, where ∇ and Δ are respectively the Levi-Civita connection and Beltrami–Laplace operator of g, and where V is a smooth vector field on $\mathcal{M}$. This operator is also a special case of V-harmonic map introduced in [12]. As in [4,11], we introduce Bakry–Emery–Ricci tensor fields

$$\mathrm{Ric}_V := \mathrm{Ric} - \frac{1}{2}\mathscr{L}_V g, \qquad \mathrm{Ric}_V^{n,m} := \mathrm{Ric}_V - \frac{1}{n-m} V \otimes V \tag{1.2}$$

for any number $n > m$, where $\mathscr{L}_V$ stands for the Lie derivative along the direction V. When $V = \nabla f$, we simply write Ric_V and $\mathrm{Ric}_V^{n,m}$ as Ric_f and $\mathrm{Ric}_f^{n,m}$ respectively.

The equation

$$\mathrm{Ric}_V = \lambda g, \quad \lambda \in \mathbf{R},$$

is exactly the Ricci soliton equation, which is one-to-one corresponding to a self-similar solution of Ricci flow (see, [13]). A basic example of Ricci solitons is Hamilton's cigar soliton or Witten's black hole, which is the complete Riemann surface $(\mathbf{R}^2, g_{cs})$ where

$$g_{cs} := \frac{dx \otimes dx + dy \otimes dy}{1 + x^2 + y^2}.$$

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It is easy to see that the scalar curvature of g_{cs} is $4/(1+x^2+y^2)$ and hence the cigar soliton is not Ricci-flat. An important result about the cigar soliton is that it is rotationally symmetric, has positive Gaussian curvature, is asymptotic to a cylinder near infinity, and, up to homothety, is the unique rotationally symmetric gradient Ricci soliton of positive curvature on $\mathbf{R}^2$. Hamilton [17] showed that any complete noncompact steady gradient Ricci soliton with positive Gaussian curvature is a cigar soliton.

To study the Ricci-flat metric on complete noncompact Riemannian manifold, the author [25] found a criterion on Ricci-flat metrics motivated from the steady gradient Ricci soliton. Moreover, the author introduced a class of Ricci flow type parabolic differential equation:

$$\partial_t g(t) = -2\mathrm{Ric}_{g(t)} + 2\alpha_1 \nabla_{g(t)}\phi(t) \otimes \nabla_{g(t)}\phi(t) + 2\alpha_2 \nabla^2_{g(t)}\phi(t), \tag{1.3}$$

$$\partial_t \phi(t) = \Delta_{g(t)}\phi(t) + \beta_1 |\nabla_{g(t)}\phi(t)|^2_{g(t)} + \beta_2\phi(t) \tag{1.4}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are given constants. Note that Eq. (1.3) can be written as

$$\partial_t g(t) = -2\mathrm{Ric}^{n,m}_{V(t)} \tag{1.5}$$

for some suitable constants α_1, α_2, n, where $V(t) := \nabla\phi(t)$. Hence the Bakry–Emery–Ricci curvature naturally appears in [25]. Under some hypotheses on initial data and constants α_i, β_i, the author proved the short time existence and Bernstein's type estimates for (1.3)–(1.4) in [25].

Another important relation between Bakry–Emery–Ricci curvature is the study of Killing vector fields. The authors in [26] investigated the gradient flow for the functional

$$\mathcal{I}(X) := \int_{\mathcal{M}} |\mathscr{L}_X g|^2 dV \tag{1.6}$$

on the space of smooth vector fields. The critical point X of $\mathcal{I}$ satisfies

$$\Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j = 0. \tag{1.7}$$

We then in [26] introduced a flow

$$\partial_t X_t = \Delta X_t + \nabla \mathrm{div}(X_t) + \mathrm{Ric}(X_t), \qquad X_0 := X, \tag{1.8}$$

to study the existence of nonzero Killing vector fields on a closed positively curved manifold. Actually, we showed that

Theorem 1.1 (*Li–Liu [26], 2011*)**.** *Suppose that $(\mathcal{M}, g)$ is a closed and orientable Riemannian manifold. If X is a smooth vector field, there exists a unique smooth solution X_t to the flow (1.8) for all time t. As t goes to infinity, the vector field X_t converges uniformly to a Killing vector field X_∞.*

The above theorem does *not* give a nontrivial Killing vector field, since Bochner's theorem implies that there is no nontrivial Killing vector field on a closed Riemannian manifold with negative Ricci curvature. For more information on the flow (1.8), we refer to the paper [26]. In the same paper [26], we give the second criterion on the existence of Killing vector fields. This observation is based on the following identity

$$\int_{\mathcal{M}} \left[(\mathscr{L}_X g)(X, X) + \frac{1}{2}\mathrm{div}(X)|X|^2 \right] dV = 0$$

where X is a smooth vector field on $\mathcal{M}$. A quite simple argument showed that

Theorem 1.2 (*Li–Liu [26], 2011*)**.** *A smooth vector field X on a closed and orientable Riemannian manifold $(\mathcal{M}, g)$ is Killing if and only if*

$$0 = \Delta X + \nabla \mathrm{div}(X) + \mathrm{Ric}_{-2X}(X) + \frac{1}{2}\mathrm{div}(X)X. \tag{1.9}$$

The third criterion in [26] is based on Lott's observation [28]:

$$\int_{\mathcal{M}} |\mathscr{L}_X g|^2 e^{-f} dV = -\int_{\mathcal{M}} \left\langle X, \Delta_f X + \nabla \mathrm{div}_f(X) + \mathrm{Ric}_f(X) \right\rangle e^f dV.$$

The we proved the following

Theorem 1.3 (*Li–Liu [26], 2011*)**.** *Given any smooth function f on a closed and orientable Riemannian manifold $(\mathcal{M}, g)$. A smooth vector field X is Killing if and only if it satisfies*

$$0 = \Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j + \nabla_j f (\mathscr{L}_X g)^{ij}. \tag{1.10}$$

In particular, X is Killing if and only if

$$0 = \Delta X^i + \nabla^i \mathrm{div}(X) + R^i{}_j X^j + \nabla_j \mathrm{div}(X)(\mathscr{L}_X g)^{ij}. \tag{1.11}$$

Those elliptic equations (1.9)–(1.10) can be made into the corresponding parabolic equations which may play well in the study of the existence of nontrivial Killing vector fields and moreover in the study of Hopf's conjecture and Yau's problem.

We now state our main results in this paper. The first three results are about Cheng–Yau estimates for complete Riemannian manifold with $\mathrm{Ric}_V^{n,m}$ bounded from below.

Theorem 1.4. *Let $(\mathcal{M}, g)$ be a compact m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -K$, where $K \geq 0$ is a constant. If u is a solution of $\Delta_V u = 0$ which is bounded from below, then*

$$|\nabla u| \leq \sqrt{(n-1)K}\left(u - \inf_{\mathcal{M}} u\right). \tag{1.12}$$

In particular, if $\mathrm{Ric}_V^{n,m} \geq 0$, then every positive solution of $\Delta_V u = 0$ must be constant.

Theorem 1.5. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -(n-1)K$ where $K \geq 0$ is a constant. If u is a positive solution of $\Delta_V u = 0$ on $\mathcal{M}$, for any $r > 0$, we have*

$$\sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leq 8(n-1)\left(\frac{1}{r} + \sqrt{K}\right). \tag{1.13}$$

Corollary 1.6. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -(n-1)K$ where $K \geq 0$ is a constant.*

(i) *If $(\mathcal{M}, g)$ is noncompact and u is a positive solution of $\Delta_V u = 0$ on $\mathcal{M}$, then*

$$\sup_{\mathcal{M}} \frac{|\nabla u|}{u} \leq 8(n-1)\sqrt{K}. \tag{1.14}$$

(ii) *If u is a solution of $\Delta_V u = 0$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x,r/2)} |\nabla u| \leq 16(n-1)\left(\frac{1}{r} + \sqrt{K}\right) \sup_{B(x,r)} |u|. \tag{1.15}$$

(iii) *If u is a positive solution of $\Delta_V u = 0$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x,r/2)} u \leq e^{8(n-1)(1+2r\sqrt{K})} \inf_{B(x,r/2)} u. \tag{1.16}$$

When $V \equiv 0$, those estimates are the classical results [10,34]. If V is gradient, the above results reduce to those of [23].

Recall that [14] a triple $(\mathcal{M}, g, \mu)$ is called a weighted Riemannian manifold, if $(\mathcal{M}, g)$ is a Riemannian manifold and μ is a measure on $\mathcal{M}$ with a smooth positive density function f (that is, $d\mu = f dV_g$). The *weighted divergence* and the *weighted Laplace operator* are defined by

$$\mathrm{div}_\mu = \frac{1}{f}\mathrm{div}(f), \qquad \Delta_\mu := \mathrm{div}_\mu \circ \nabla$$

respectively, where ∇ is the Levi-Civita connection of g. There are two examples of Δ_μ:

(a) When $V = \nabla f$, the operator Δ_V is exactly the weighted Laplace operator of the weighted Riemannian manifold $(\mathcal{M}, g, \mu)$ where $(\mu = e^f dV_g)$. Indeed,

$$\Delta_\mu = \frac{1}{e^f}\mathrm{div}(e^f \nabla) = \frac{1}{e^f}\left(e^f \Delta + \langle \nabla e^f, \nabla \rangle\right) = \Delta + \langle \nabla f, \nabla \rangle =: \Delta_f.$$

(b) In [31], the authors introduced a diffusion-type operator

$$L = \frac{1}{B}\mathrm{div}(A\nabla)$$

where A, B are some sufficiently smooth positive functions on $\mathcal{M}$. Set

$$\tilde{g} := \frac{B}{A} g, \qquad d\tilde{\mu} := B dV_g.$$

Then L is the weighted Laplace operator of the weighted Riemannian manifold $(\mathcal{M}, \tilde{g}, \tilde{\mu})$ since

$$\widetilde{\Delta}_{\tilde{\mu}} = \mathrm{div}_{\tilde{\mu}} \circ \widetilde{\nabla} = \frac{1}{B}\mathrm{div}\left(B\frac{A}{B}\nabla\right) = L.$$

In both cases, Δ_f or L can be viewed as the special case of Δ_V on some Riemannian manifold.

Theorem 1.7. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -(n-1)K(1+d^2)^{\delta/2}$, where $K \geq 0$, $\delta < 4$, and d denotes the distance function from a fixed point. If $F \in C^1(\mathbf{R})$ and $u \in C^3(\mathcal{M})$ is a global solution of*

$$\Delta_V u = F(u)$$

with

$$|u| \leq D(1+d)^\nu, \qquad F'(u) \geq (n-1)K(1+d^2)^{\delta/2}$$

on $\mathcal{M}$ for some constants $D > 0$ and $0 < \nu < \min\{1, 1-\frac{\delta}{4}\}$, then u must be constant.

Theorem 1.7 generalized the similar result in [30,31]. The proof is based on variants of V-Bochner–Weitzenböck formula stated in Section 2.

Next three estimates are about Li–Yau gradient estimates for positive solutions of weighted heat type equation on a complete Riemannian manifold, and extend the corresponding results in [42] from heat type equation to weighted heat type equation.

Theorem 1.8. *Let $(\mathcal{M}, g)$ be a compact m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq 0$. Suppose that the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ is convex whenever $\partial\mathcal{M} \neq \emptyset$. Let u be a positive solution of*

$$(\Delta_V - \partial_t)\, u = au \ln u$$

on $\mathcal{M} \times (0, T]$ for some constant a, with Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\mathcal{M} \times (0, T]$.

(1) *If $q \leq 0$ then*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a\ln u \leq \frac{n}{2t} - \frac{na}{2}$$

on $\mathcal{M} \times (0, T]$.

(2) *If $a \geq 0$ then*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a\ln u \leq \frac{n}{2t}.$$

Theorem 1.9. *Let $(\mathcal{M}, g)$ be a complete manifold with boundary $\partial\mathcal{M}$. Assume that $p \in \mathcal{M}$ and the geodesic ball $B(p, 2R)$ does not intersect $\partial\mathcal{M}$. We denote by $-K(2R)$ with $K(2R) \geq 0$, a lower bound of $\mathrm{Ric}_V^{n,m}$ on the ball $B(p, 2R)$. Let q be a function defined on $\mathcal{M} \times [0, T]$ which is C^2 in the x variable and C^1 in the t variable. Assume that*

$$\Delta_V q \leq \theta(2R), \qquad |\nabla q| \leq \gamma(2R)$$

on $B(p, 2R) \times [0, T]$ for some constants $\theta(2R)$ and $\gamma(2R)$. If u is a positive solution of the equation

$$(\Delta_V - q - \partial_t)\, u = au \ln u$$

on $\mathcal{M} \times (0, T]$ for some constant a, then for any $\alpha > 1$ and $\epsilon \in (0, 1)$, on $B(p, R)$, u satisfies the following estimates:

(1) *for $a \geq 0$, we have*

$$\begin{aligned} |\nabla f|^2 - \alpha f_t - \alpha q - \alpha a f \;\leq\; & \frac{n\alpha^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\alpha^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} \\ & + \frac{n\alpha^2[K + a(\alpha-1)]}{(1-\epsilon)(\alpha-1)} + \left(\frac{[\alpha\theta + (\alpha-1)\gamma]n\alpha^2}{2(1-\epsilon)}\right)^{1/2}. \end{aligned}$$

(2) *for $a \leq 0$, we have*

$$\begin{aligned} |\nabla f|^2 - \alpha f_t - \alpha q - \alpha a f \;\leq\; & \frac{n\alpha^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\alpha^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} \\ & + \frac{n\alpha^2\left[K - \frac{a}{2}a(\alpha-1)\right]}{(1-\epsilon)(\alpha-1)} + \left(\frac{[\alpha\theta + (\alpha-1)\gamma]n\alpha^2}{2(1-\epsilon)}\right)^{1/2}. \end{aligned}$$

Here $f := \ln u$ and $A = [2C_1^2 + (n-1)C_1^2(1+R\sqrt{K}) + C_2]/R^2$ for some positive constants C_1, C_2.

Corollary 1.10. *If $(\mathcal{M}, g)$ is a complete noncompact Riemannian manifold without boundary and $\mathrm{Ric}_V^{n,m} \geq -K$ on $\mathcal{M}$, then any positive solution u of the equation*

$$\partial_t u = \Delta_V u$$

on $\mathcal{M} \times (0, T]$ satisfies

$$\frac{|\nabla u|^2}{u^2} - \alpha \frac{u_t}{u} \le \frac{n\alpha^2 K}{\alpha - 1} + \frac{n\alpha^2}{2t} \tag{1.17}$$

for any $\alpha > 1$.

As pointed in [34], the estimate (1.17) still holds for any closed Riemannian manifold with $\mathrm{Ric}_V^{n,m} \ge -K$.

Thirdly, we derive Hamilton's Harnack inequality for Δ_V operator. Setting $V \equiv 0$ in Theorem 1.11, we obtain the classical result of Hamilton [16]. Later Kotschwar [21] extended Hamilton's gradient estimate to complete noncompact Riemannian manifold. Li [24] proved Hamilton's gradient estimate for Δ_V where $V = -\nabla\phi$, both in compact case and noncompact case.

Theorem 1.11. *Suppose that $(\mathcal{M}, g)$ is a compact Riemannian manifold with $\mathrm{Ric}_V \ge -K$ where $K \ge 0$. If u is a solution of $\partial_t u = \Delta_V u$ with $0 < u \le A$ on $\mathcal{M} \times (0, T]$, then*

$$\frac{|\nabla u|^2}{u^2} \le \left(\frac{2K}{e^{2Kt} - 1} + 2K \right) \ln \frac{A}{u} \le \left(\frac{1}{t} + 2K \right) \ln \frac{A}{u} \tag{1.18}$$

on $\mathcal{M} \times (0, T]$.

As a consequence of Theorem 1.11, we generalize a result in [7,24] about the Liouville theorem.

Corollary 1.12. *Suppose that $(\mathcal{M}, g)$ is a compact Riemannian manifold with $\mathrm{Ric}_V \ge -K$ where $K \ge 0$. If u is a positive solution of $\Delta_V u = 0$ on $\mathcal{M}$ then*

$$|\nabla \ln u|^2 \le 2K \ln \frac{\sup_{\mathcal{M}} u}{u}. \tag{1.19}$$

In particular if $\mathrm{Ric}_V \ge 0$ every bounded solution u satisfying $\Delta_V u = 0$ must be constant.

A local version of Hamilton's estimate was proved by Souplet and Zhang [35] for Δ, while by Arnaudon, Thalmaier, and Wang [2] for the general operator Δ_V. A probabilistic proof of Hamilton's estimates for Δ and Δ_V with $V = -\nabla\phi$ can be found in [1,24]. In this paper we give a geometric proof of Hamilton's estimate for Witten's Laplacian, following the method in [21] together with Karp–Li–Grigor'yan maximum principle for complete manifolds.

Theorem 1.13. *Suppose that $(\mathcal{M}, g)$ is a complete noncompact Riemannian manifold with $\mathrm{Ric}_f^{n,m} \ge -K$ where $K \ge 0$. If u is a solution of $\partial_t u = \Delta_f u$ with $0 < u \le A$ on $\mathcal{M} \times (0, T]$, then*

$$\frac{|\nabla u|^2}{u^2} \le \left(\frac{2K}{e^{2Kt} - 1} + 2K \right) \ln \frac{A}{u} \le \left(\frac{1}{t} + 2K \right) \ln \frac{A}{u} \tag{1.20}$$

on $\mathcal{M} \times (0, T]$.

We compare other Hamilton's estimates with (1.20). In our geometric proof we require the curvature condition $\mathrm{Ric}_f^{n,m} \ge -K$ in order to use the Bakry–Qian's Laplacian comparison theorem without any additional requirement on the potential function f. If we use the curvature condition $\mathrm{Ric}_f \ge -K$ in our geometric proof, then some conditions on f would be required (see [11,37]). A probabilistic proof of Li [24] shows a similar estimate

$$\frac{|\nabla u|^2}{u^2} \le \left(\frac{2}{t} + 2K \right) \ln \frac{A}{u}$$

where $0 < u \le A$ on $\mathcal{M} \times (0, T]$ and $\mathrm{Ric}_f \ge -K$.

In the last part, we generalize Hessian estimates for positive solutions of the heat equation in [18] to these of the weighted heat equation.

Theorem 1.14. *Let $(\mathcal{M}, g)$ be a closed m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \ge -K$ where $K \ge 0$.*

(a) *If u is a solution of $\partial_t u = \Delta_V u$ in $\mathcal{M} \times (0, T]$ and $0 < u \le A$, then*

$$\nabla^2 u \le \left(B + \frac{5}{t} \right) u \left(1 + \ln \frac{A}{u} \right) g \tag{1.21}$$

in $\mathcal{M} \times (0, T]$, where $B = 10m^{3/2} n \mathcal{K}_V$,

$$\mathcal{K}_V := K_1 + K_2 + \sqrt{(K_1 + K_2)K + K_2 + K_1 \sup_{\mathcal{M}} |V|^2}$$

with $K_1 = \max_{\mathcal{M}}(|\mathrm{Rm}| + |\mathrm{Ric}_V|)$ and $K_2 = \max_{\mathcal{M}} |\nabla \mathrm{Ric}_V|$.

(b) *If u is a solution of $\partial_t u = \Delta_V u$ in $Q_{R,T}(x_0, t_0)$ and $0 < u \le A$, then*

$$\nabla^2 u \le C_1 \left(\frac{1}{T} + \frac{1 + R\sqrt{K}}{R^2} + B \right) u \left(1 + \ln \frac{A}{u} \right)^2 g \tag{1.22}$$

in $Q_{R/2,T/2}(x_0, t_0)$, where $B = C_2 m^{5/2} n^2 \mathcal{K}_V$ and C_1, C_2 are positive universal constants.

2. *V*-Bochner–Weitzenböck formula and its applications

To prove Li–Yau–Hamilton estimates for V-weighted equation, we need the following Bochner–Weitzenböck formula for V-Laplace operator.

Lemma 2.1. *Given a smooth vector field V on a Riemannian manifold $(\mathcal{M}, g)$. For any smooth function u on $\mathcal{M}$, we have*

$$\frac{1}{2}\Delta_V |\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle. \tag{2.1}$$

In particular, we have

$$\frac{1}{2}\Delta_V |\nabla u|^2 \ge \frac{1}{n}(\Delta_V u)^2 + \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle\,, \tag{2.2}$$

$$\frac{1}{2}\Delta_V |\nabla u|^2 \ge |\nabla^2 u|^2 + \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle, \tag{2.3}$$

$$\frac{1}{2}\Delta_V |\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle + \frac{\langle V, \nabla u \rangle^2}{n-m} \tag{2.4}$$

for any $n > m$.

Proof. When $V = \nabla f$ for some smooth function f, this inequality was established by many authors (e.g., [23]). The proof is based on the usual Bochner–Weitzenböck formula

$$\frac{1}{2}\Delta |\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle. \tag{2.5}$$

By definition, it follows that

$$\begin{aligned}
\frac{1}{2}\Delta_V |\nabla u|^2 &= \frac{1}{2}\Delta |\nabla u|^2 + \frac{1}{2}\langle V, \nabla |\nabla u|^2 \rangle \\
&= |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla \Delta u, \nabla u \rangle + \frac{1}{2}\langle V, \nabla |\nabla u|^2 \rangle.
\end{aligned}$$

The last two terms of the right-hand side becomes

$$\begin{aligned}
\langle \nabla \Delta u, \nabla u \rangle + \frac{1}{2}\langle V, \nabla |\nabla u|^2 \rangle &= \langle \nabla(\Delta_V u - \langle V, \nabla u \rangle), \nabla u \rangle + V^i \nabla^i u \nabla_i \nabla_j u \\
&= \langle \nabla \Delta_V u, \nabla u \rangle - \nabla^i u \nabla_i (V^j \nabla_j u) + V^i \nabla^j u \nabla_i \nabla_j u \\
&= \langle \nabla \Delta_V u, \nabla u \rangle - \nabla_i u \nabla_j u \nabla^i V^j \\
&= \langle \nabla \Delta_V u, \nabla u \rangle - \nabla_i u \nabla_j u \left(\frac{\nabla^i V^j + \nabla^j V^i}{2} \right) \\
&= \langle \nabla \Delta_V u, \nabla u \rangle - \frac{1}{2}\mathscr{L}_V g(\nabla u, \nabla u).
\end{aligned}$$

Therefore

$$\frac{1}{2}\Delta_V |\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle.$$

This is the identity (2.1), which implies (2.4) and (2.3). From the elementary inequality $m|\nabla^2 u|^2 \ge |\Delta u|^2$ we arrive at

$$\frac{1}{2}\Delta_V |\nabla u|^2 \ge \frac{1}{m}|\Delta u|^2 + \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u \rangle + \frac{1}{n-m}\langle V, \nabla u \rangle^2$$

for any $n > m$. Using another elementary inequality

$$(a-b)^2 \ge \frac{1}{t}a^2 - \frac{1}{t-1}b^2, \quad t > 1,$$

we get

$$\begin{aligned}\frac{1}{m}|\Delta u|^2 &= \frac{1}{m}\left(\Delta_V u - \langle V, \nabla u\rangle\right)^2\\ &\geq \frac{1}{m}\left(\frac{1}{n/m}(\Delta_V u)^2 - \frac{1}{n/m-1}\langle V, \nabla u\rangle^2\right)\\ &= \frac{1}{n}(\Delta_V u)^2 - \frac{1}{n-m}\langle V, \nabla u\rangle^2.\end{aligned}$$

Together those inequalities, we obtain the desired inequality (2.2). □

Corollary 2.2. *Let u be a solution of $\Delta_V u = 0$ and $n > m$ a constant. Then*

$$|\nabla u|\Delta_V|\nabla u| \geq \frac{1}{n-1}|\nabla(|\nabla u|)|^2 + \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u). \tag{2.6}$$

Proof. From the identity

$$\Delta_V|\nabla u|^2 = 2|\nabla u|\Delta_V|\nabla u| + 2|\nabla(|\nabla u|)|^2$$

and the above lemma, we obtain

$$|\nabla u|\Delta_V|\nabla u| = |\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) \tag{2.7}$$

for any solution u of $\Delta_V u = 0$. Now the proof follows from the similar argument as stated in [34,41,23]. For the completeness, we present it here. Given any point $p \in \mathscr{M}$ and choose a normal coordinate system $(x^1, \ldots, x^m)$ at p so that $u_i(p) = |\nabla u|(p)$ and $u_i(p) = 0$ for all $2 \leq i \leq m$, where $u_i := \partial u/\partial x^i$, etc. Then

$$|\nabla(|\nabla u|)|^2 = \sum_{1\leq j\leq m} u_{1j}^2.$$

Since $0 = \Delta u + \langle V, u\rangle$ it follows that

$$-\sum_{2\leq i\leq m} u_{ii} = u_{11} + V_1 u_1$$

and then, for any $\alpha > 0$, (see page 1310–1311 in [23] for some detail)

$$\begin{aligned}|\nabla^2 u|^2 - |\nabla(|\nabla u|)|^2 &\geq \sum_{2\leq i\leq m} u_{i1}^2 + \frac{1}{m-1}(u_{11} + V_1 u_1)^2\\ &\geq \left(\sum_{2\leq i\leq m} u_{i1}^2 + \frac{1}{(1+\alpha)(m-1)}u_{11}^2\right) - \frac{1}{\alpha(m-1)}|V_1 u_1|^2\\ &\geq \frac{1}{(1+\alpha)(m-1)}|\nabla(|\nabla u|)|^2 - \frac{1}{\alpha(m-1)}|\langle V, \nabla u\rangle|^2.\end{aligned}$$

Consequently,

$$|\nabla u|\Delta_V|\nabla u| \geq \frac{1}{(1+\alpha)(m-1)}|\nabla(|\nabla u|)|^2 + \left(\mathrm{Ric}_V - \frac{1}{\alpha(m-1)}V\otimes V\right)(\nabla u, \nabla u).$$

Taking $\alpha = \frac{n-m}{m-1}$ yields the desired result. □

Theorem 2.3. *Let $(\mathscr{M}, g)$ be a compact m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -K$, where $K \geq 0$ is a constant. If u is a solution of $\Delta_V u = 0$ which is bounded from below, then*

$$|\nabla u| \leq \sqrt{(n-1)K}\left(u - \inf_{\mathscr{M}} u\right). \tag{2.8}$$

In particular, if $\mathrm{Ric}_V^{n,m} \geq 0$, then every positive solution of $\Delta_V u = 0$ must be constant.

Proof. By replacing u by $u - \inf_{\mathscr{M}} u$, we may assume that u is positive. The proof is similar to that in [41,34,23]. Let $\phi := |\nabla u|/u = |\nabla \ln u|$. Then

$$\nabla\phi = \frac{\nabla|\nabla u|}{u} - \frac{|\nabla u|\nabla u}{u^2}.$$

At any point where $\nabla u \neq 0$, using

$$\Delta_V|\nabla u| = u\Delta_V\phi + 2\langle\nabla\phi, \nabla u\rangle + \phi\Delta_V u = u\Delta_V\phi + 2\langle\nabla\phi, \nabla u\rangle$$

we obtain

$$\begin{aligned}\Delta_V\phi &= \frac{\Delta_V|\nabla u|}{u} - \frac{2\langle\nabla\phi, \nabla u\rangle}{u}\\ &\geq \frac{1}{u|\nabla u|}\left(\frac{1}{n-1}|\nabla(|\nabla u|)|^2 - K|\nabla u|^2\right) - \frac{2\langle\nabla\phi, \nabla u\rangle}{u}\\ &= \frac{1}{n-1}\frac{|\nabla(|\nabla u|)|^2}{u|\nabla u|} - K\phi - \frac{2\langle\nabla\phi, \nabla u\rangle}{u}.\end{aligned}$$

As [34,23], we furthermore get the following inequality

$$\Delta_V\phi \geq -K\phi - \left(2 - \frac{2}{n-1}\right)\frac{\langle\nabla\phi, \nabla u\rangle}{u} + \frac{1}{n-1}\phi^3.$$

If ϕ achieves its maximum at some point $p \in \mathcal{M}$, then $\nabla\phi = \Delta\phi = 0$ at p and $\Delta_V\phi(p) \leq 0$. Plugging this into the above inequality implies $\phi(p) \leq \sqrt{(n-1)K}$ and hence $|\nabla u| \leq \sqrt{(n-1)K}u$ on $\mathcal{M}$. □

Using Lemma 2.1, Bakry and Qian [5] studied the eigenvalue problem of Δ_V.

3. Bakry–Qian's comparison theorem

If $\mathrm{Ric}_V^{n,m} \geq K$ for some constant K, then the elliptic operator Δ_V satisfies the $CD(K, n)$ condition in the sense of Bakry [3], see also [6,23]. Bakry and Qian proved the following Laplacian comparison theorem for Δ_V.

Theorem 3.1 (*Bakry–Qian [6]*). *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold and $\mathrm{Ric}_V^{n,m} \geq (n-1)K$, where $K = K(d(p))$ is a function depending on the distance function $d(p) = d(p, p_0)$ for a fixed point $p_0 \in \mathcal{M}$. Let θ_K be the solution defined on the maximal interval $(0, \delta_K)$ of the Riccati equation*

$$\dot{\theta}_K(r) = -K(r) - \theta_K^2(r), \qquad \lim_{r\to 0} r\theta_K(r) = n-1, \tag{3.1}$$

and δ_K is the explosion time of θ_K such that

$$\lim_{r\to\delta_K-}\theta_K(r) = -\infty.$$

Then

(i) *If $\delta_K < \infty$, then $\mathcal{M}$ is compact and the diameter of $(\mathcal{M}, g)$ is bounded from above by δ_K.*
(ii) *For any $p \in \mathcal{M} \setminus \mathrm{cut}(p_0)$, we have*

$$\Delta_V d \leq (n-1)\theta_K(d). \tag{3.2}$$

(iii) *We denote by μ_V an invariant measure for Δ_V, that is a solution of $\Delta_V^*(\mu_V) = 0$. By ellipticity, such an invariant measure has a smooth density with respect to dV_g. Then the Laplacian comparison theorem holds in the sense of distributions:*

$$\int_{\mathcal{M}} d(\Delta_V^*\varphi)\,d\mu_V \leq \int_{\mathcal{M}} \varphi(m-1)\theta_K(d)\,d\mu_V \tag{3.3}$$

for any nonnegative smooth function φ on $\mathcal{M}$ with compact support.

Compared with the space-form, we obtain

Corollary 3.2. *If $(\mathcal{M}, g)$ is a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq (n-1)K$, where $K \in \mathbf{R}$, and if $p \in \mathcal{M}$, then for any $x \in \mathcal{M}$ where $d(x) := d(x, p)$ is smooth, we have*

$$\Delta_V d \leq \begin{cases}(n-1)\sqrt{K}\cot\left(\sqrt{K}d\right), & K > 0,\\ \dfrac{n-1}{d}, & K = 0,\\ (n-1)\sqrt{|K|}\coth\left(\sqrt{|K|}d\right), & K < 0.\end{cases} \tag{3.4}$$

Using $x\coth x \leq 1 + x$ yields (see also [6,33])

Corollary 3.3. *If $(\mathcal{M}, g)$ is a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq (n-1)K$, where $K \leq 0$, then*

$$\Delta_V d \leq \frac{n-1}{d} + (n-1)\sqrt{|K|} \tag{3.5}$$

in the sense of distributions. In particular, if $(\mathcal{M}, g)$ is a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq 0$, then

$$d\Delta_V d \leq n-1 \tag{3.6}$$

in the sense of distributions.

Theorem 3.4. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -(n-1)K$ where $K \geq 0$ is a constant. If u is a positive solution of $\Delta_V u = 0$ on $\mathcal{M}$, then*

$$\sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leq 8(n-1)\left(\frac{1}{r} + \sqrt{K}\right). \tag{3.7}$$

Proof. Recall

$$\Delta_V \phi \geq -(n-1)K\phi - \left(2 - \frac{2}{n-1}\right)\frac{\langle \nabla\phi, \nabla u\rangle}{u} + \frac{1}{n-1}\phi^3, \quad \phi := \frac{|\nabla u|}{u}.$$

For any $r > 0$, we consider the quantity

$$F(y) := (r^2 - d^2(x,y))\phi(y), \quad y \in B(x,r).$$

It is clear that

$$\nabla F = -\phi\Delta(d^2) + (r^2-d^2)\nabla\phi, \qquad \Delta_V F = (r^2-d^2)\Delta_V\phi - \phi\Delta_V(d^2) - 2\langle\nabla(d^2), \nabla\phi\rangle.$$

Now the proof of the above estimate is similar to Theorem 3.1 (page 19–20) in [34] or Theorem 2.3 (page 1313–1314) in [23]. Since $F = 0$ on the boundary of $B(x,r)$, if $|\nabla u| \neq 0$, then F must achieve its maximum at some $x_0 \in B(x,r)$. By Calabi's argument [8,10,34], we may assume that x_0 is not a cut point of x. Then F is smooth near x_0 and hence

$$\Delta F \leq 0 = \nabla F \quad \text{at } x_0.$$

It follows that $\Delta_V F(x_0) = \Delta F(x_0) + \langle V, \nabla F\rangle(x_0) \leq 0$ and then

$$\frac{\nabla\phi}{\phi} = \frac{\nabla(d^2)}{r^2-d^2}, \qquad \frac{\Delta_V\phi}{\phi} - \frac{\Delta_V(d^2)}{r^2-d^2} - \frac{2\langle\nabla(d^2), \nabla\phi\rangle}{\phi(r^2-d^2)} \leq 0 \quad \text{at } x_0.$$

Consequently,

$$\frac{\Delta_V\phi}{\phi} - \frac{\Delta_V(d^2)}{r^2-d^2} - \frac{2|\nabla(d^2)|^2}{(r^2-d^2)^2} \leq 0 \quad \text{at } x_0.$$

By (3.5) we have

$$\Delta_V(d^2) = 2d\Delta_V d + 2|\nabla d|^2 \leq 2 + 2(n-1)(1+\sqrt{K}d)$$

so that, using $|\nabla(d^2)|^2 = 4d^2$,

$$\begin{aligned} 0 &\geq \frac{\Delta_V\phi}{\phi} - \frac{2+2(n-1)(1+\sqrt{K}d)}{r^2-d^2} - \frac{8d^2}{(r^2-d^2)^2} \\ &\geq -(n-1)K - \left(2-\frac{2}{n-1}\right)\frac{\langle\nabla\phi, \nabla u\rangle}{\phi u} + \frac{1}{n-1}\phi^2 - \frac{2+2(n-1)(1+\sqrt{K}d)}{r^2-d^2} - \frac{8d^2}{(r^2-d^2)^2} \end{aligned}$$

at x_0. On the other hand,

$$\frac{\langle\nabla\phi, \nabla u\rangle}{\phi u} = \left\langle \frac{\nabla\phi}{\phi}, \frac{\nabla u}{u}\right\rangle = \frac{\langle\nabla(d^2), \nabla u\rangle}{(r^2-d^2)u} = \frac{2d\langle\nabla d, \nabla u\rangle}{(r^2-d^2)u} \leq \frac{2d}{r^2-d^2}\phi.$$

Therefore

$$0 \geq \frac{1}{n-1}F^2 - \frac{4(n-2)}{n-1}dF - [2+2(n-1)(1+\sqrt{K}d)](r^2-d^2) - 8d^2 - (n-1)K(r^2-d^2)^2$$

at x_0. When $n = 2$, the above inequality becomes

$$F \leq \sqrt{Kr^4 + (12+2\sqrt{K}r)r^2} \leq \sqrt{12}r(1+\sqrt{K}r).$$

When $n \geq 3$, we arrive at

$$\frac{1}{n-1}F^2 - \frac{4(n-2)}{n-1}rF \leq [2+2(n-1)(1+\sqrt{K}r)]r^2 + 8r^2 + (n-1)Kr^4$$

and hence

$$\begin{aligned} F(x_0) &\leq r\left[2(n-2)+(n-1)\sqrt{(\sqrt{K}r)^2+2\sqrt{K}r+6+\frac{2(n+1)}{(n-1)^2}}\right] \\ &\leq r\left[2(n-2)+(n-1)\sqrt{8}(1+\sqrt{K}r)\right] \\ &\leq 4\sqrt{2}(n-1)r(1+\sqrt{K}r). \end{aligned}$$

In both cases, we obtain

$$F \leq 4\sqrt{2}(n-1)r(1+\sqrt{K}r) \quad \text{on } B(x,r).$$

In particular

$$\frac{3}{4}r^2 \sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leq \sup_{B(x,r/2)} F \leq 4\sqrt{2}(n-1)r(1+\sqrt{K}r)$$

which implies

$$\sup_{B(x,r/2)} \frac{|\nabla u|}{u} \leq \frac{16\sqrt{2}}{3}(n-1)\left(\frac{1}{r}+\sqrt{K}\right) \leq 8(n-1)\left(\frac{1}{r}+\sqrt{K}\right).$$

This is the desired estimate. □

As an immediate consequence, we have the following variants corollaries parallel to these in [34,23].

Corollary 3.5. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -(n-1)K$ where $K \geq 0$ is a constant.*

(i) *If $(\mathcal{M}, g)$ is noncompact and u is a positive solution of $\Delta_V u = 0$ on $\mathcal{M}$, then*

$$\sup_{\mathcal{M}} \frac{|\nabla u|}{u} \leq 8(n-1)\sqrt{K}. \tag{3.8}$$

(ii) *If u is a solution of $\Delta_V u = 0$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x,r/2)} |\nabla u| \leq 16(n-1)\left(\frac{1}{r}+\sqrt{K}\right) \sup_{B(x,r)} |u|. \tag{3.9}$$

(iii) *If u is a positive solution of $\Delta_V u = 0$ on a geodesic ball $B(x, r)$, then*

$$\sup_{B(x,r/2)} u \leq e^{8(n-1)(1+2r\sqrt{K})} \inf_{B(x,r/2)} u. \tag{3.10}$$

4. A generalized diffusion operator

Recall that a triple $(\mathcal{M}, g, \mu)$ is called a *weighted Riemannian manifold* (for more detail, see [14]), if $(\mathcal{M}, g)$ is a Riemannian manifold and μ is a measure on $\mathcal{M}$ with a smooth positive density function f (that is, $d\mu = f dV_g$). The *weighted divergence* and the *weighted Laplace operator* are defined by

$$\mathrm{div}_\mu = \frac{1}{f}\mathrm{div}(f), \qquad \Delta_\mu := \mathrm{div}_\mu \circ \nabla$$

respectively, where ∇ is the Levi-Civita connection of g. There are two examples of Δ_μ:

(a) When $V = \nabla f$, the operator Δ_V is exactly the weighted Laplace operator of the weighted Riemannian manifold $(\mathcal{M}, g, \mu)$ where $(\mu = e^f dV_g)$. Indeed,

$$\Delta_\mu = \frac{1}{e^f}\mathrm{div}(e^f \nabla) = \frac{1}{e^f}\left(e^f \Delta + \langle \nabla e^f, \nabla \rangle\right) = \Delta + \langle \nabla f, \nabla \rangle =: \Delta_f.$$

(b) In [31], the authors introduced a diffusion-type operator

$$L = \frac{1}{B}\mathrm{div}(A\nabla)$$

where A, B are some sufficiently smooth positive functions on $\mathcal{M}$. Set

$$\tilde{g} := \frac{B}{A} g, \qquad d\tilde{\mu} := B\, dV_g.$$

Then L is the weighted Laplace operator of the weighted Riemannian manifold $(\mathcal{M}, \tilde{g}, \tilde{\mu})$ since

$$\widetilde{\Delta}_{\tilde{\mu}} = \mathrm{div}_{\tilde{\mu}} \circ \widetilde{\nabla} = \frac{1}{B} \mathrm{div}\left(B \frac{A}{B} \nabla\right) = L.$$

In both cases, Δ_f or L can be viewed as the special case of Δ_V on some Riemannian manifold. In this section we study the following diffusion Poisson equation

$$\Delta_V u = F(u) \tag{4.1}$$

on a complete noncompact m-dimensional Riemannian manifold $\mathcal{M}$, where $m \geq 2$. Let $B(p, r)$ denote the geodesic ball of radius $r > 0$ centered at p and $d(x) := \mathrm{dist}_g(x, p)$.

Lemma 4.1. *Let $\mathrm{Ric}_V^{n,m} \geq -(n-1)K$ on $B(p, r)$, where $K \geq 0$ is a constant and $n > m$, and $u \in C^3(\mathcal{M})$ is a solution of $\Delta_V u = F(u)$ on $\mathcal{M}$ for some $F \in C^1(\mathbf{R})$. Consider the function*

$$H(x) = [r^2 - d^2(x)]^2 |\nabla u|^2(x) G[u(x)] \tag{4.2}$$

where $G \in C^2(\mathbf{R})$ and $G(u) > 0$ on $B(p, r)$. Then

$$\begin{aligned}
&\Delta_V \ln H + \left\langle \nabla \ln H, \nabla \ln H + \frac{8d\nabla d}{r^2 - d^2} - \frac{2G'(u)}{G(u)} \nabla u \right\rangle \\
&\quad \geq -2(n-1)K + 2F'(u) + \frac{G'(u)}{G(u)} F(u) + \frac{2G(u)G''(u) - 3G'(u)^2}{2G(u)^2} |\nabla u|^2 \\
&\qquad - \frac{4dG'(u)}{(r^2 - d^2)G(u)} |\nabla u| - \frac{4[n + (n-1)\sqrt{K}d]}{r^2 - d^2} - \frac{16d^2}{(r^2 - d^2)^2},
\end{aligned}$$

and

$$\begin{aligned}
&\Delta_V \ln H + 2\left\langle \nabla \ln H, \nabla \ln H + \frac{8d\nabla d}{r^2 - d^2} - \frac{2G'(u)}{G(u)} \nabla u \right\rangle \\
&\quad \geq -2(n-1)K + 2F'(u) + \frac{8G(u)G''(u) - (8+n)G''(u)^2}{8G(u)^2} |\nabla u|^2 \\
&\qquad - \frac{8dG'(u)}{(r^2 - d^2)G(u)} |\nabla u| - \frac{4[n + (n-1)d\sqrt{K}]}{r^2 - d^2} - \frac{24d^2}{(r^2 - d^2)^2}
\end{aligned}$$

on points where H is positive.

Proof. On points where H is positive, we get

$$\begin{aligned}
\nabla \ln H &= \frac{\nabla H}{H} = \frac{G'(u)}{G(u)} \nabla u + \frac{\nabla |\nabla u|^2}{|\nabla u|^2} - \frac{2\nabla(d^2)}{r^2 - d^2}, \\
\Delta_V \ln H &= \frac{\Delta_V H}{H} - |\nabla \ln H|^2 \\
&= -2\frac{\Delta_V (d^2)}{r^2 - d^2} + \frac{\Delta_V |\nabla u|^2}{|\nabla u|^2} + \frac{G'(u)}{G(u)} \Delta_V u - 2\frac{|\nabla(d^2)|^2}{(r^2 - d^2)^2} + \frac{G(u)G''(u) - G'(u)^2}{G(u)^2} |\nabla u|^2 - \frac{|\nabla |\nabla u|^2|^2}{|\nabla u|^4}.
\end{aligned}$$

By (2.3) and Kato's inequality

$$|\nabla |\nabla u|^2|^2 \leq 4|\nabla u|^2 |\nabla^2 u|^2,$$

we arrive at

$$\frac{\Delta_V |\nabla u|^2}{|\nabla u|^2} \geq \frac{|\nabla |\nabla u|^2|^2}{2|\nabla u|^4} - 2(n-1)K + 2F'(u).$$

Using the facts $\Delta_V(d^2) \leq 2 + 2(n-1)(1 + \sqrt{K}d)$ and $|\nabla(d^2)|^2 = 4d^2$ yields

$$\begin{aligned}
\Delta_V \ln H &\geq -2(n-1)K + 2F'(u) + \frac{G'(u)}{G(u)} F(u) - \frac{|\nabla |\nabla u|^2|^2}{2|\nabla u|^4} \\
&\quad + \frac{G(u)G''(u) - G'(u)^2}{G(u)^2} |\nabla u|^2 - \frac{4[n + (n-1)d\sqrt{K}]}{r^2 - d^2} - \frac{8d^2}{(r^2 - d^2)^2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{|\nabla|\nabla u|^2|^2}{2|\nabla u|^4} &= \frac{1}{2}\left(\nabla \ln H + \frac{2\nabla(d^2)}{r^2-d^2} - \frac{G'(u)}{G(u)}\nabla u\right)^2 \\ &= \frac{G'(u)^2}{2G(u)^2}|\nabla u|^2 + \frac{8d^2}{(r^2-d^2)^2} - \frac{4dG'(u)}{(r^2-d^2)G(u)}\langle \nabla u, \nabla d\rangle + (\nabla \ln H)^2 + \left\langle \nabla \ln H, \frac{8d\nabla d}{r^2-d^2} - \frac{2G'(u)}{G(u)}\nabla u\right\rangle\end{aligned}$$

which implies the following inequality

$$\begin{aligned}&\Delta_V \ln H + \left\langle \nabla \ln H, \nabla \ln H + \frac{8d\nabla d}{r^2-d^2} - \frac{2G'(u)}{G(u)}\nabla u\right\rangle \\ &\geq -2(n-1)K + 2F'(u) + \frac{G'(u)}{G(u)}F(u) + \frac{2G(u)G''(u) - 3G'(u)^2}{2G(u)^2}|\nabla u|^2 \\ &\quad - \frac{4dG'(u)}{(r^2-d^2)G(u)}|\nabla u| - \frac{4[n+(n-1)\sqrt{K}d]}{r^2-d^2} - \frac{16d^2}{(r^2-d^2)^2}.\end{aligned}$$

Recall the formula proved in Lemma 2.1

$$\frac{1}{2}\Delta_V|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u) + \langle \nabla \Delta_V u, \nabla u\rangle + \frac{1}{n-m}\langle V, \nabla u\rangle^2.$$

Therefore

$$\frac{\Delta_V|\nabla u|^2}{|\nabla u|^2} \geq 2\frac{|\nabla^2 u|^2}{|\nabla u|^2} - 2(n-1)K + 2F'(u) + \frac{2}{n-m}\frac{\langle V, \nabla u\rangle^2}{|\nabla u|^2}.$$

As in [31], we have

$$\frac{|\nabla^2 u|^2}{|\nabla u|^2} \geq \frac{1}{m|\nabla u|^2}\left(\frac{(\Delta_V u)^2}{1+\gamma} - \frac{\langle V, \nabla u\rangle^2}{\gamma}\right)$$

for any $\gamma > 0$, and hence

$$\frac{\Delta_V|\nabla u|^2}{|\nabla u|^2} \geq -2(n-1)K + 2F'(u) - \frac{G'(u)}{G(u)}\Delta_V u - \frac{n}{8}\frac{G'(u)^2}{G(u)^2}|\nabla u|^2$$

by taking $\gamma = \frac{n-m}{m}$. Consequently

$$\begin{aligned}&\Delta_V \ln H + 2\left\langle \nabla \ln H, \nabla \ln H + \frac{8d\nabla d}{r^2-d^2} - \frac{2G'(u)}{G(u)}\nabla u\right\rangle \\ &\geq -2(n-1)K + 2F'(u) + \frac{8G(u)G''(u) - (8+n)G''(u)^2}{8G(u)^2}|\nabla u|^2 \\ &\quad - \frac{8dG'(u)}{(r^2-d^2)G(u)}|\nabla u| - \frac{4[n+(n-1)d\sqrt{K}]}{r^2-d^2} - \frac{24d^2}{(r^2-d^2)^2}. \quad \square\end{aligned}$$

It is observed that the above lemma is similar to that in [30, Lemma 1.2, page 14]. As a consequence we have

Theorem 4.2. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -(n-1)K(1+d^2)^{\delta/2}$, where $K \geq 0$, $\delta < 4$, and d denotes the distance function from a fixed point. If $F \in C^1(\mathbf{R})$ and $u \in C^3(\mathcal{M})$ is a global solution of*

$$\Delta_V u = F(u)$$

with

$$|u| \leq D(1+d)^{\nu}, \qquad F'(u) \geq (n-1)K(1+d^2)^{\delta/2}$$

on $\mathcal{M}$ for some constants $D > 0$ and $0 < \nu < \min\{1, 1-\frac{\delta}{4}\}$, then u must be constant.

5. Li–Yau–Hamilton estimates

In this section we consider the following parabolic equation

$$(\Delta_V - q - \partial_t)\, u = au \ln u \tag{5.1}$$

on $\mathcal{M} \times (0, T]$, where a is a constant and $q \in C^2(\mathcal{M} \times (0, T])$. When $V \equiv 0$ or V is gradient, this equation was considered in [38,42]. Suppose that u is a positive solution of (5.1) and consider

$$f := \ln u. \tag{5.2}$$

Then (5.1) can be rewritten as

$$(\Delta_V - \partial_t) f = -|\nabla f|^2 + q + af. \tag{5.3}$$

Lemma 5.1. *Let $(\mathcal{M}, g)$ be a complete m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -K$, where K is a nonnegative function on $\mathcal{M}$. If f is a solution of (5.3), then the quantity*

$$F := t(|\nabla f|^2 - \alpha f_t - \alpha q - \alpha af), \quad \alpha \geq 1 \tag{5.4}$$

satisfies

$$\begin{aligned}(\Delta_V - \partial_t) F \geq{} & -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af\right)^2 \\ & - \alpha t \Delta_V g - 2(\alpha-1)t\langle \nabla f, \nabla q\rangle - 2(\alpha-1)ta|\nabla f|^2 + \alpha at\left(|\nabla f|^2 - q - f_t - af\right).\end{aligned}$$

Proof. By the linearity, we have

$$\Delta_V F = t\Delta_V|\nabla f|^2 - \alpha t \Delta_V f_t - \alpha t \Delta_V g - \alpha a t \Delta_V f.$$

Using Lemma 2.1, together with

$$\Delta_V f = -|\nabla f|^2 + q + f_t + af = -\frac{F}{t} - (\alpha-1)(q + f_t + af),$$

we arrive at

$$\begin{aligned}\Delta_V F \geq{} & \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af\right)^2 - 2t\left\langle \nabla f, \nabla\left(\frac{F}{t} + (\alpha-1)(q + f_t + af)\right)\right\rangle \\ & - 2Kt|\nabla f|^2 - t\alpha\left(-\frac{F}{t} - (\alpha-1)(q + f_t + af)\right)_t - \alpha t\Delta_V g - \alpha a t \Delta_V f \\ ={} & \frac{2t}{n}\left(|\nabla f|^2 - q - f_t - af\right)^2 - 2\langle \nabla f, \nabla F\rangle - 2(\alpha-1)t\langle \nabla f, \nabla f_t\rangle \\ & - 2(\alpha-1)t\langle \nabla f, \nabla q\rangle - 2(\alpha-1)ta|\nabla f|^2 - 2Kt|\nabla f|^2 + \alpha F_t \\ & - \alpha\left(|\nabla f|^2 - \alpha f_t - \alpha q - \alpha af\right) + \alpha(\alpha-1)tq_t + \alpha(\alpha-1)tf_{tt} + \alpha(\alpha-1)taf_t - \alpha t \Delta_V q - \alpha a t \Delta_V f.\end{aligned}$$

On the other hand,

$$F_t = |\nabla f|^2 - \alpha f_t - \alpha q - \alpha af + t\left(\partial_t|\nabla f|^2 - \alpha f_{tt} - \alpha q_t - \alpha a f_t\right).$$

This implies the result. □

Theorem 5.2. *Let $(\mathcal{M}, g)$ be a compact m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq 0$. Suppose that the boundary $\partial\mathcal{M}$ of $\mathcal{M}$ is convex whenever $\partial\mathcal{M} \neq \emptyset$. Let u be a positive solution of*

$$(\Delta_V - \partial_t)\, u = au \ln u$$

on $\mathcal{M} \times (0, T]$ for some constant a, with Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial\mathcal{M} \times (0, T]$.

(1) *If $q \leq 0$ then*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leq \frac{n}{2t} - \frac{na}{2}$$

on $\mathcal{M} \times (0, T]$.

(2) *If $a \geq 0$ then*

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leq \frac{n}{2t}.$$

Proof. From Lemma 5.1 we obtain

$$\begin{aligned}(\Delta_V - \partial_t) F &\geq -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2t}{n}(|\nabla f|^2 - f_t - af)^2 + at(|\nabla f|^2 - f_t - af)\\ &= -2\langle \nabla f, \nabla F\rangle - \frac{F}{t} + \frac{2F^2}{nt} + aF\\ &= -2\langle \nabla f, \nabla F\rangle + \frac{2F}{nt}\left(F - \frac{n}{2} + \frac{ant}{2}\right)\end{aligned}$$

where $F = t(|\nabla f|^2 - f_t - af)$ and $f = \ln u$.

Now the proof is similar to that in [27,42]. For convenience, we give some detail here. Firstly we assume $a \leq 0$. In this case we claim that $F \leq \frac{n}{2} - \frac{ant}{2}$ on $\mathcal{M} \times (0, T]$. Otherwise

$$F(x_0, t_0) = \sup_{\mathcal{M}\times(0,T]} F > \frac{n}{2} - \frac{ant}{2} \geq \frac{n}{2} > 0$$

for some point $(x_0, t_0) \in \mathcal{M} \times (0, T]$, hence $t_0 > 0$. If x_0 is an interior point of $\mathcal{M}$, then $\Delta F(x_0, t_0) \leq \nabla F(x_0, t_0) = 0 \leq F_t(x_0, t_0)$. Consequently,

$$\Delta_V F(x_0, t_0) = \Delta F(x_0, t_0) + \langle V, \nabla F\rangle(x_0, t_0) \leq 0.$$

At the point (x_0, t_0) we get

$$0 \geq \frac{2F}{nt}\left(F - \frac{n}{2} + \frac{ant}{2}\right)$$

from which $F(x_0, t_0) \leq \frac{n}{2} - \frac{ant_0}{2}$. This contradiction implies that $F \leq \frac{n}{2} - \frac{ant}{2}$ on $\mathcal{M} \times (0, T]$. Next we consider the case that x_0 is on the boundary of $\mathcal{M}$. The strong maximum principle shows that $\frac{\partial F}{\partial \nu}(x_0, t_0) > 0$. Choose an orthonormal basis $(e_i)_{1\leq i\leq m}$ for $T\mathcal{M}$, where $e_m := \partial/\partial\nu$. Compute

$$F_\nu = 2t \sum_{1\leq j\leq m-1} f_j f_{j\nu} + 2t f_\nu f_{\nu\nu} - f_{t\nu} - a f_\nu.$$

Since $u_\nu = 0$ on $\partial\mathcal{M}$, it follows that $f_\nu = 0$ on $\partial\mathcal{M}$ and hence

$$F_\nu = 2\sum_{1\leq j\leq m-1} f_j f_{j\nu} = -2t \sum_{1\leq j,k\leq m-1} h_{jk} f_j f_k = -2t\mathrm{II}(\nabla f, \nabla f)$$

because $f_{j\nu} = -\sum_{1\leq k\leq m-1} h_{jk} f_k$, where h_{jk} are components of the second fundamental form II of $\partial\mathcal{M}$. Consequently $\mathrm{II}(\nabla f, \nabla f)(x_0, t_0) < 0$ which contradicts the convexity of $\partial\mathcal{M}$. Hence $F \leq \frac{n}{2} - \frac{ant}{2}$.

We now consider the rest case $a \geq 0$. Since $n/2t > 0$, we may assume that $F \geq 0$. In this case we obtain

$$(\Delta_V - \partial_t) F \geq -2\langle \nabla f, \nabla F\rangle + \frac{2F}{nt}\left(F - \frac{n}{2}\right)$$

which reduces to [27] and by the same computation we can conclude that $F \leq n/2$. □

Theorem 5.3. *Let $(\mathcal{M}, g)$ be a complete manifold with boundary $\partial\mathcal{M}$. Assume that $p \in \mathcal{M}$ and the geodesic ball $B(p, 2R)$ does not intersect $\partial\mathcal{M}$. We denote by $-K(2R)$ with $K(2R) \geq 0$, a lower bound of $\mathrm{Ric}_V^{n,m}$ on the ball $B(p, 2R)$. Let q be a function defined on $\mathcal{M} \times [0, T]$ which is C^2 in the x variable and C^1 in the t variable. Assume that*

$$\Delta_V q \leq \theta(2R), \qquad |\nabla q| \leq \gamma(2R)$$

on $B(p, 2R) \times [0, T]$ for some constants $\theta(2R)$ and $\gamma(2R)$. If u is a positive solution of the equation

$$(\Delta_V - q - \partial_t) u = au \ln u$$

on $\mathcal{M} \times (0, T]$ for some constant a, then for any $\alpha > 1$ and $\epsilon \in (0, 1)$, on $B(p, R)$, u satisfies the following estimates:

(1) *for $a \geq 0$, we have*

$$\begin{aligned}|\nabla f|^2 - \alpha f_t - \alpha q - \alpha af \leq{}& \frac{n\alpha^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\alpha^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2}\\ &+ \frac{n\alpha^2[K + a(\alpha-1)]}{(1-\epsilon)(\alpha-1)} + \left(\frac{[\alpha\theta + (\alpha-1)\gamma]n\alpha^2}{2(1-\epsilon)}\right)^{1/2}.\end{aligned}$$

(2) *for* $a \leq 0$, *we have*

$$|\nabla f|^2 - \alpha f_t - \alpha q - \alpha af \;\leq\; \frac{n\alpha^2}{2(1-\epsilon)t} + \frac{(A+\gamma)n\alpha^2}{2(1-\epsilon)} + \frac{n^2\beta^4 C_1^2}{4\epsilon(1-\epsilon)(\beta-1)R^2} + \frac{n\alpha^2\left[K - \frac{a}{2}a(\alpha-1)\right]}{(1-\epsilon)(\alpha-1)} + \left(\frac{[\alpha\theta + (\alpha-1)\gamma]n\alpha^2}{2(1-\epsilon)}\right)^{1/2}.$$

Here $f := \ln u$ *and* $A = [2C_1^2 + (n-1)C_1^2(1+R\sqrt{K}) + C_2]/R^2$ *for some positive constants* C_1, C_2.

Proof. Set $F := t(|\nabla f|^2 - \alpha f_t - \alpha q - \alpha af)$. As in [9,27,29,34,42], we choose a smooth function $\tilde{\varphi}(r)$ defined on $[0,\infty)$ such that

$$\tilde{\varphi}(r) = \begin{cases} 1, & r \in [0,1], \\ 0, & r \in [2,\infty), \end{cases}$$

and

$$-C_1 \leq \tilde{\varphi}'(r)\varphi^{-1/2}(r) \leq 0, \qquad \tilde{\varphi}(r) \geq -C_2$$

for some positive constants C_1, C_2. Set

$$\varphi(x) := \tilde{\varphi}\left(\frac{1}{R}d(x)\right)$$

where $r(x)$ denotes the distance function from p to x. By Calabi's trick (see, e.g., [8,10,34]), we may assume that the function φ is smooth in the ball $B(p, 2R)$. By Corollary 3.3, we obtain

$$\frac{|\nabla\varphi|^2}{\varphi} \leq \frac{C_1^2}{R^2}, \qquad \Delta_V\varphi \geq -\frac{(n-1)C_1(1+R\sqrt{K}) + C_2}{R^2}.$$

Now the proof is similar to that in [42]; we present the detail here for completeness. From Lemma 5.1, we arrive at

$$\begin{aligned}
\Delta_V(\varphi F) &= F\Delta_V\varphi + 2\langle\nabla\varphi, \nabla F\rangle + \varphi\Delta_V F \\
&\geq -F\left[\frac{2C_1^2 + (n-1)C_1(1+R\sqrt{K}) + C_2}{R^2}\right] + \frac{2}{\varphi}\langle\nabla\varphi, \nabla(\varphi F)\rangle \\
&\quad + \varphi\Big[F_t - 2\langle\nabla f, \nabla F\rangle - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - f_t - q - af\right)^2 \\
&\quad - \alpha t\Delta_V q - 2(\alpha-1)t\langle\nabla f, \nabla q\rangle - 2(\alpha-1)ta|\nabla f|^2 + \alpha at\left(|\nabla f|^2 - f_t - q - af\right)\Big].
\end{aligned}$$

Fix a time $T' \leq T$ and consider a point $(x_0, t_0) \in \mathcal{M} \times [0, T']$ where φF achieves its maximum. Without loss of generality, we may assume that $(\varphi F)(x_0, t_0) > 0$ (so that $t_0 > 0$), otherwise it is clear. Since

$$\Delta(\varphi F)(x_0, t_0) \leq 0 = \nabla(\varphi F)(x_0, t_0) \leq (\varphi F)_t(x_0, t_0),$$

it follows that

$$\Delta_V(\varphi F)(x_0, t_0) = \Delta(\varphi F)(x_0, t_0) + \langle V, \nabla(\varphi F)\rangle(x_0, t_0) \leq 0.$$

Letting

$$A := \frac{2C_1^2 + (n-1)C_1(1+R\sqrt{K}) + C_2}{R^2}$$

and noting that $\varphi\nabla F = -F\nabla\varphi$ at the point (x_0, t_0), we obtain

$$\begin{aligned}
0 \;\geq\; & -AF + 2F\langle\nabla f, \nabla\varphi\rangle - \frac{\varphi F}{t_0} - 2Kt_0\varphi|\nabla f|^2 + \frac{2t_0}{n}\varphi\left(|\nabla f|^2 - f_t - q - af\right)^2 \\
& - \alpha t_0\varphi\Delta_V q - 2(\alpha-1)t_0\varphi\langle\nabla f, \nabla q\rangle - 2(\alpha-1)t_0 a\varphi|\nabla f|^2 + \alpha at_0\varphi\left(|\nabla f|^2 - f_t - q - af\right)
\end{aligned}$$

at the point (x_0, t_0). As in [9,40,42], set

$$\mu := \frac{|\nabla f|^2}{F}(x_0, t_0) \geq 0.$$

Then

$$\begin{aligned}|\nabla f|^2 - f_t - q - af &= F\mu + \frac{1}{\alpha}\left(\frac{F}{t_0} - |\nabla f|^2\right)\\ &= F\mu + \frac{F}{t_0\alpha} - \frac{\mu F}{\alpha} = F\left(\mu - \frac{\mu t_0 - 1}{\alpha t_0}\right)\end{aligned}$$

$$\langle \nabla f, \nabla\varphi\rangle \le |\nabla f|\,|\nabla\varphi| \le \frac{C_1}{R}\varphi^{1/2}|\nabla f|$$

at the point (x_0, t_0). Setting $G := \varphi F$ and using the above inequalities we arrive at

$$\begin{aligned}At_0G \ge& -\frac{2C_1t_0}{R}\mu^{1/2}G^{3/2} - \varphi G + \frac{2}{n\alpha^2}[1 + (\alpha-1)\mu t_0]^2G^2\\ &- 2\varphi t_0^2[K + a(\alpha-1)]\mu G + a\varphi t_0[1 + (\alpha-1)\mu t_0]G - \alpha(\varphi t_0)^2\theta - 2(\alpha-1)t_0^2\varphi^{3/2}\gamma\mu^{1/2}G^{1/2}\end{aligned}$$

at the point (x_0, t_0). For any $\epsilon \in (0, 1)$, we have the following elementary inequality

$$\frac{2C_1t_0}{R}\mu^{1/2}G^{3/2} \le \frac{2\epsilon}{n\alpha^2}[1 + (\alpha-1)\mu t_0]^2G^2 + \frac{n\alpha^2C_1^2t_0^2\mu G}{2\epsilon R^2[1 + (\alpha-1)\mu t_0]^2},$$

which, together with $2\mu^{1/2}G^{1/2} \le 1 + \mu G$, implies that

$$\begin{aligned}\frac{2(1-\epsilon)[1 + (\alpha-1)\mu t_0]^2G^2}{n\alpha^2} \le& \Bigg[At_0 + \varphi + \frac{n\alpha^2C_1^2t_0^2\mu}{2\epsilon R^2[1 + (\alpha-1)\mu t_0]^2}\\ &+ 2\varphi t_0^2[K + a(\alpha-1)]\mu - a\varphi t_0[1 + (\alpha-1)\mu t_0] + (\alpha-1)t_0^2\varphi^{3/2}\gamma\mu\Bigg]G\\ &+ [\alpha\varphi^2\theta + (\alpha-1)\varphi^{3/2}\gamma]t_0^2\end{aligned}$$

at the point (x_0, t_0). Note that $0 \le \varphi \le 1$ and $1 + (\alpha-1)\mu t_0 \ge 1$. Therefore the above inequality reduces to the following

$$\begin{aligned}\frac{2(1-\epsilon)G^2}{n\alpha^2} \le& \left[At_0 + 1 + \frac{n\alpha^2C_1^2t_0}{2\epsilon R^2(\alpha-1)} + \frac{2\varphi t_0[K + a(\alpha-1)]\mu t_0}{[1 + (\alpha-1)\mu t_0]^2} - \frac{a\varphi t_0}{1 + (\alpha-1)\mu t_0} + \gamma t_0\right]G\\ &+ [\alpha\theta + (\alpha-1)\gamma]t_0^2\end{aligned}$$

at the point (x_0, t_0). Now the desired result follows by using the fact that

$$x \le \frac{aq}{2} + \sqrt{b + \left(\frac{a}{2}\right)^2} \le \frac{a}{2} + \sqrt{b} + \frac{a}{2} = a + \sqrt{b}$$

whenever $x^2 \le ax + b$ for some $a, b, x \ge 0$. For example, when $a \le 0$, we obtain

$$\begin{aligned}G^2 \le& \left[\frac{An\alpha^2t_0}{2(1-\epsilon)} + \frac{n\alpha^2}{2(1-\epsilon)} + \frac{n^2\alpha^4C_1^2t_0}{4\epsilon(1-\epsilon)R^2(\alpha-1)} + \frac{n\alpha^2[K + a(\alpha-1)]t_0}{(1-\epsilon)(\alpha-1)} + \frac{n\alpha^2\gamma t_0}{2(1-\epsilon)}\right]G\\ &+ \frac{[\alpha\theta + (\alpha-1)\gamma]n\alpha^2t_0^2}{2(1-\epsilon)}\end{aligned}$$

at the point (x_0, t_0), which yields an upper bound for G given by

$$G \le \left[\frac{(A+\gamma)n\alpha^2}{2(1-\epsilon)} + \frac{n^2\alpha^4C_1^2}{4\epsilon(1-\epsilon)(\alpha-1)R^2} + \frac{n\alpha^2[K + a(\alpha-1)]}{(1-\epsilon)(\alpha-1)}\right]T' + \left(\frac{[\alpha\theta + (\alpha-1)\gamma]n\alpha^2}{2(1-\epsilon)}\right)^{1/2}T' + \frac{n\alpha^2}{2(1-\epsilon)}$$

at the point (x_0, t_0). By the construction of φ, we have $F \le G(x_0, t_0)$ on $B(p, R) \times [0, T']$. Since T' was arbitrary, it proves (1). Similarly, one can get the desired result in (2). $\square$

Corollary 5.4. *If $(\mathcal{M}, g)$ is a complete noncompact Riemannian manifold without boundary and $\mathrm{Ric}_V^{n,m} \ge -K$ on $\mathcal{M}$, then any positive solution u of the equation*

$$\partial_t u = \Delta_V u$$

on $\mathcal{M} \times (0, T]$ satisfies

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u} \le \frac{n\alpha^2K}{\alpha-1} + \frac{n\alpha^2}{2t} \tag{5.5}$$

for any $\alpha > 1$.

Remark 5.5. As pointed in [34], the estimate (5.5) still holds for any closed Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -K$.

Next we derive Hamilton's Harnack inequality for weighted heat equation. Let u be a positive solution of $\partial_t u = \Delta_V u$.

Lemma 5.6. *Suppose $(\mathscr{M}, g)$ is a compact Riemannian manifold. We have*

$$(\partial_t - \Delta_V)\,\frac{|\nabla u|^2}{u} = -\frac{2}{u}\left[\left|\nabla^2 u - \frac{1}{u}\nabla u \otimes \nabla u\right|^2 + \mathrm{Ric}_V(\nabla u, \nabla u)\right]. \tag{5.6}$$

When $V \equiv 0$ this identity is due to the classical result proved by Hamilton [16]. Li [24] generalized this identity to the Witten Laplacian $L = \Delta_V$, where $V = -\nabla\phi$ for some C^2-function ϕ on $\mathscr{M}$.

Proof. As in [16,24], we directly compute the evolution equation for $|\nabla u|^2/u$ as follows. Since $\partial_t u = \Delta_V u$, it follows that

$$\begin{aligned}\partial_t\left(\frac{|\nabla u|^2}{u}\right) &= \frac{\partial_t|\nabla u|^2}{u} - \frac{|\nabla u|^2}{u^2}\partial_t u\\ &= \frac{2}{u}\langle\nabla u, \nabla\partial_t u\rangle - \frac{|\nabla u|^2}{u^2}\Delta_V u\\ &= \frac{2}{u}\langle\nabla u, \nabla\Delta_V u\rangle - \frac{|\nabla u|^2}{u^2}\Delta_V u.\end{aligned}$$

By the commutative formula $\nabla_i\Delta u = \Delta\nabla_i u - R_{ij}\nabla^j u$ we obtain

$$\begin{aligned}\nabla_i\Delta_V u &= \nabla_i\Delta u + \nabla_i(V^j\nabla_j u)\\ &= \Delta\nabla_i u - R_{ij}\nabla^j u + V^j\nabla_i\nabla_j u + \nabla_i V_j\nabla^j u\\ &= \Delta_V\nabla_i u - R_{ij}\nabla^j u + \nabla_i V_j\nabla^j u.\end{aligned}$$

Plugging this into $\partial_t(|\nabla u|^2/u)$ yields

$$\partial_t\left(\frac{|\nabla u|^2}{u}\right) = \frac{2}{u}\left[\langle\nabla u, \Delta_V\nabla u\rangle - \mathrm{Ric}(\nabla u, \nabla u) + \nabla_i V_j\nabla^i u\nabla^j u\right] - \frac{|\nabla u|^2}{u^2}\Delta_V u.$$

Because the term $\nabla^i V_j\nabla^i u\nabla_j$ is symmetric in the indices i, j, we can rewrite it as

$$\nabla_i V_j\nabla^i u\nabla^j u = \frac{1}{2}(\nabla_i V_j + \nabla_j V_i)\nabla^i u\nabla^j u = \frac{1}{2}\mathscr{L}_V g(\nabla u, \nabla u).$$

Consequently

$$\partial_t\left(\frac{|\nabla u|^2}{u}\right) = \frac{2}{u}\langle\nabla u, \Delta_V\nabla u\rangle - \frac{2}{u}\mathrm{Ric}_V(\nabla u, \nabla u) - \frac{|\nabla u|^2}{u^2}\Delta_V u.$$

Similarly, we compute

$$\Delta_V\left(\frac{|\nabla u|^2}{u}\right) = \frac{\Delta_V|\nabla u|^2}{u} + |\nabla u|^2\Delta_V(u^{-1}) + 2\langle\nabla(u^{-1}), \nabla|\nabla u|^2\rangle.$$

Using

$$\begin{aligned}\Delta_V(u^{-1}) &= \Delta(u^{-1}) + \langle V, \nabla(u^{-1})\rangle\\ &= -\frac{\Delta u}{u^2} - \frac{2|\nabla u|^2}{u^3} - \frac{\langle V, \nabla u\rangle}{u^2}\\ &= -\frac{1}{u^2}\Delta_V u + \frac{2|\nabla u|^2}{u^3},\\ 2\langle\nabla(u^{-1}), \nabla|\nabla u|^2\rangle &= -\frac{2}{u^2}\langle\nabla u, \nabla|\nabla u|^2\rangle\\ &= -\frac{4}{u^2}\nabla_i\nabla_j u\nabla^i u\nabla^j u,\end{aligned}$$

we get

$$\Delta_V\left(\frac{|\nabla u|^2}{u}\right) = \frac{2}{u}\langle\nabla u, \Delta_V\nabla u\rangle + \frac{2}{u}|\nabla^2 u|^2 - \frac{|\nabla u|^2}{u^2}\Delta_V u + \frac{2|\nabla u|^4}{u^3} - \frac{4}{u^2}\nabla_i\nabla_j u\nabla^i u\nabla^j u$$

which, together with $\partial_t(|\nabla u|^2/u)$, implies

$$(\partial_t - \Delta_V)\left(\frac{|\nabla u|^2}{u}\right) = -\frac{2}{u}\mathrm{Ric}_V(\nabla u, \nabla u) - \frac{2}{u}\left[|\nabla^2 u|^2 + \frac{|\nabla u|^4}{u^2} - \frac{2}{u}\nabla_i\nabla_j u\nabla^i u\nabla^j u\right].$$

Squaring the last term on the right-hand side we obtain the desired identity. □

Theorem 5.7. *Suppose that $(\mathcal{M}, g)$ is a compact Riemannian manifold with $\mathrm{Ric}_V \geq -K$ where $K \geq 0$. If u is a solution of $\partial_t u = \Delta_V u$ with $0 < u \leq A$ on $\mathcal{M} \times (0, T]$, then*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{2K}{e^{2Kt}-1} + 2K\right)\ln\frac{A}{u} \leq \left(\frac{1}{t} + 2K\right)\ln\frac{A}{u} \tag{5.7}$$

on $\mathcal{M} \times (0, T]$.

Proof. It follows from the above lemma that

$$(\partial_t - \Delta_V)\left(\frac{|\nabla u|^2}{u}\right) \leq \frac{2K}{u}|\nabla u|^2.$$

On the other hand, we claim that

$$(\partial_t - \Delta_V)\left(u\ln\frac{A}{u}\right) = \frac{|\nabla u|^2}{u}.$$

In fact,

$$\partial_t\left(u\ln\frac{A}{u}\right) = \ln\frac{A}{u}\partial_t u - u\frac{\partial_t u}{u} = \Delta_V u\left(\ln\frac{A}{u} - 1\right),$$

$$\begin{aligned}\Delta_V\left(u\ln\frac{A}{u}\right) &= \ln\frac{A}{u}\Delta_V u + u\Delta_V(\ln A - \ln u) + 2\left\langle\nabla u, -\frac{\nabla u}{u}\right\rangle \\ &= \ln\frac{A}{u}\Delta_V u - u\Delta_V\ln u - 2\frac{|\nabla u|^2}{u} \\ &= \ln\frac{A}{u}\Delta_V u - u\left(\frac{\Delta_V u}{u} - \frac{|\nabla u|^2}{u^2}\right) - 2\frac{|\nabla u|^2}{u} \\ &= \Delta_V u\left(\ln\frac{A}{u} - 1\right) - \frac{|\nabla u|^2}{u}.\end{aligned}$$

Choose a time-depending function φ with $\varphi(0) = 0$ and consider

$$F := \varphi\frac{|\nabla u|^2}{u} - u\ln\frac{A}{u}.$$

Therefore F satisfies the following inequality

$$(\partial_t - \Delta_V)F \leq (\varphi' + 2K\varphi - 1)\frac{|\nabla u|^2}{u}.$$

If φ is chosen so that $\varphi' + 2K\varphi - 1 \leq 0$, then $\partial_t F \leq \Delta_V F$ on $\mathcal{M} \times (0, T]$. By a maximum principle (e.g., see Theorem 4.2 in [13]), $F \leq 0$ on $\mathcal{M} \times (0, T]$ because $F(x, 0) \leq 0$ for all $x \in \mathcal{M}$. Solving the evolution inequality of φ we see that

$$\varphi(t) \leq \frac{1 - e^{-2Kt}}{2K} = \frac{e^{2Kt} - 1}{2Ke^{2Kt}}.$$

Since $e^{2Kt} \geq 1 + 2Kt$, it follows that $\frac{t}{1+2Kt} \leq \frac{e^{2Kt}-1}{2Ke^{2Kt}}$. Hence we may choose $\varphi(t) = \frac{t}{1+2Kt}$. □

As a consequence of Theorem 5.7, we generalize a result in [7,24] about the Liouville theorem.

Corollary 5.8. *Suppose that $(\mathcal{M}, g)$ is a compact Riemannian manifold with $\mathrm{Ric}_V \geq -K$ where $K \geq 0$. If u is a positive solution of $\Delta_V u = 0$ on $\mathcal{M}$ then*

$$|\nabla\ln u|^2 \leq 2K\ln\frac{\sup_{\mathcal{M}} u}{u}. \tag{5.8}$$

In particular if $\mathrm{Ric}_V \geq 0$ every bounded solution u satisfying $\Delta_V u = 0$ must be constant.

Proof. For any $x \in \mathcal{M}$ and $t > 0$, consider the function $u(x,t) := u(x)$. Then $\partial_t u = \Delta_V u$. From (5.7) we obtain

$$|\nabla \ln u|^2 \le \left(\frac{2K}{e^{2Kt}-1} + 2K\right) \ln \frac{\sup_{\mathcal{M}} u}{u};$$

letting $t \to \infty$ implies that $|\nabla \ln u|^2 \le 2K \ln(\sup_{\mathcal{M}} u/u)$.

In general, let u be any bounded solution of $\Delta_V u = 0$. For any given positive number $\epsilon > 0$, replacing u by $u - \inf_{\mathcal{M}} u + \epsilon$ in (5.8) we arrive at

$$\left|\nabla \ln\left(u - \inf_{\mathcal{M}} u + \epsilon\right)\right|^2 \le 2K \ln \frac{\sup_{\mathcal{M}} u - \inf_{\mathcal{M}} u + \epsilon}{u - \inf_{\mathcal{M}} u + \epsilon}.$$

When $K = 0$, this inequality shows that $|\nabla \ln(u - \inf_{\mathcal{M}} u + \epsilon)|^2 = 0$ on $\mathcal{M}$ which means that $u - \inf_{\mathcal{M}} u + \epsilon$ is a constant C_ϵ. Thus u must be $\inf_{\mathcal{M}} u$ a constant. $\square$

Setting $V \equiv 0$ in Theorem 5.7, we obtain the classical result of Hamilton [16]. Later Kotschwar [21] extended Hamilton's gradient estimate to complete noncompact Riemannian manifold. Li [24] proved Hamilton's gradient estimate for Δ_V where $V = -\nabla\phi$, both in compact case and noncompact case. A local version of Hamilton's estimate was proved by Souplet and Zhang [35] for Δ, while by Arnaudon, Thalmaier, and Wang [2] for the general operator Δ_V. A probabilistic proof of Hamilton's estimates for Δ and Δ_V with $V = -\nabla\phi$ can be found in [1,24]. In this paper we give a geometric proof of Hamilton's estimate for Witten's Laplacian, following the method in [21] together with Karp–Li–Grigor'yan maximum principle for complete manifolds. In an unpublished paper [20], Karp and Li established a maximum principle for complete manifolds (see also [21,22,32]), which was independently found by Grigor'yan [15] with a slightly weaker condition. Actually, Grigor'yan proved this type of maximum principle for complete weighted manifolds [15,14].

Theorem 5.9 (*Karp–Li–Grigor'yan*)**.** *Let $(\mathcal{M}, g, e^f dV)$ be a complete weighted manifold, and let $u(x,t)$ be a solution of*

$$\partial_t u \le \Delta_f u \quad \text{in } \mathcal{M} \times (0,T], \qquad u(\cdot,0) \le 0.$$

Assume that for some $x_0 \in \mathcal{M}$ and for all $r > 0$,

$$\int_0^T \int_{B(x_0,r)} u_+^2(x,t) e^{f(x)} dV(x) dt \le e^{\alpha(r)}$$

where $u_+ := \max\{u, 0\}$ and $\alpha(r)$ is a positive increasing function on $(0,\infty)$ such that

$$\int_0^\infty \frac{r}{\alpha(r)} dr = \infty.$$

Then $u \le 0$ on $\mathcal{M} \times (0,T]$.

The proof can be found in [14, Theorem 11.9], where the author proved the result for $\partial_u = \Delta_f u$ with $u(\cdot,0) = 0$, however, the proof still works for the above setting without any changes.

Theorem 5.10. *Suppose that $(\mathcal{M}, g)$ is a complete noncompact Riemannian manifold with $\mathrm{Ric}_f^{n,m} \ge -K$ where $K \ge 0$. If u is a solution of $\partial_t u = \Delta_f u$ with $0 < u \le A$ on $\mathcal{M} \times (0,T]$, then*

$$\frac{|\nabla u|^2}{u^2} \le \left(\frac{2K}{e^{2Kt}-1} + 2K\right) \ln \frac{A}{u} \le \left(\frac{1}{t} + 2K\right) \ln \frac{A}{u} \tag{5.9}$$

on $\mathcal{M} \times (0,T]$.

Proof. We follow the method in [21]. Given any positive number $\epsilon > 0$, consider $u_\epsilon := u + \epsilon$ and

$$F_\epsilon := \varphi \frac{|\nabla u_\epsilon|^2}{u_\epsilon} - u_\epsilon \ln \frac{A_\epsilon}{u_\epsilon}$$

where $A_\epsilon := A + \epsilon$, $\varphi(0) = 0$, and $\varphi' + 2K\varphi - 1 \le 0$. Since $\partial_t u_\epsilon = \Delta_f u_\epsilon$, it follows from the computation in Theorem 5.7 we have

$$\left(\partial_t - \Delta_f\right) F_\epsilon \le 0, \qquad F_\epsilon(\cdot,0) \le 0, \qquad (F_\epsilon)_+ \le \frac{\varphi}{\epsilon} |\nabla u_\epsilon|^2.$$

Let us estimate

$$\int_0^T \int_{B(x_0,r)} \left(\frac{\varphi}{\epsilon} |\nabla u_\epsilon|^2\right)^2 e^f dV dt.$$

As pointed out in the proof of Theorem 5.7, we have chosen $\varphi(t) = (1 - e^{-2Kt})/2K$. We need the following

Proposition 5.11. *Suppose that* $(\mathcal{M}, g)$ *is a complete noncompact Riemannian manifold with* $\mathrm{Ric}_V^{n,m} \geq -K$ *where* $K \geq 0$*. If* u *is a solution of* $\partial_t u = \Delta_V u$ *with* $0 < u \leq A$ *on* $\mathcal{M} \times (0, T]$*, then for any* $a > 2$ *we have*

$$\varphi|\nabla u|^2 \leq \frac{(a+1)^3 A^2}{2a(a-2)} \left\{ 1 + \left(1 - e^{-2Kt}\right) \left[\frac{1}{a} + C \frac{1 + (n-1)(1 + r\sqrt{K})}{2Kr^2} \right] \right\} \tag{5.10}$$

on $B(x_0, r) \times [0, T]$ *for some positive constant* C*. In particular,*

$$\varphi|\nabla u|^2 \leq \frac{(a+1)^3}{2a^2(a-2)} \left(a + 1 - e^{-2Kt}\right) A^2 \tag{5.11}$$

on $\mathcal{M} \times (0, T]$ *for any* $a > 2$*.*

Proof. Compute

$$\begin{aligned} \partial_t |\nabla u|^2 &= 2\langle \nabla u, \nabla \Delta_V u \rangle \\ &= 2\nabla^i u \left(\Delta_V \nabla_i u - R_{ij} \nabla^j u + \nabla_i V^j \nabla_j u \right) \\ &= \Delta_V |\nabla u|^2 - 2|\nabla^2 u|^2 - 2\mathrm{Ric}_V(\nabla u, \nabla u), \end{aligned}$$

$$\partial_t u^2 = \Delta_V u^2 - 2|\nabla u|^2.$$

Consider the quantity

$$G := (aA^2 + u^2)|\nabla u|^2, \quad a > 0,$$

which satisfies the following evolution equation

$$(\partial_t - \Delta_V) G = -2|\nabla u|^4 - 2(aA^2 + u^2) \left[|\nabla^2 u|^2 + \mathrm{Ric}_V(\nabla u, \nabla u) \right] - 8u \nabla_i \nabla_j u \nabla^i \nabla^j u.$$

From the Cauchy inequality, we have $8u\nabla_i\nabla_j u \nabla^i u \nabla^j u \leq \eta |\nabla u|^4 + \frac{16}{\eta} u^2 |\nabla^2 u|^2$ for any $\eta > 0$, and hence

$$\begin{aligned} (\partial_t - \Delta_V) G &\leq (\eta - 2)|\nabla u|^4 + \left[\frac{16}{\eta} - 2(1+a) \right] u^2 |\nabla^2 u|^2 + 2(1+a)KA^2 |\nabla u|^2 \\ &= \frac{4 - 2a}{1+a} \left(\frac{G}{aA^2 + u^2} \right)^2 + \frac{2(1+a)A^2}{aA^2 + u^2} KG \\ &\leq -2 \frac{a-2}{(a+1)^3} \frac{G^2}{A^4} + 2 \frac{a+1}{a} KG \end{aligned}$$

where we chosen $\eta = \frac{8}{1+a}$ in the second step and $a > 2$ in the third step. Here we used a fact that

$$\mathrm{Ric}_V(\nabla u, \nabla u) = \mathrm{Ric}_V^{n,m}(\nabla u, \nabla u) + \frac{\langle V, \nabla u \rangle^2}{n - m} \geq -K.$$

As in the proof of Theorem 5.3, we take a smooth function χ equal to 1 on $B(x_0, r)$ and supported in $B(x_0, 2r)$, satisfying

$$\frac{|\nabla \chi|^2}{\chi} \leq \frac{C_1^2}{r^2}, \qquad \Delta_V \chi \geq -\frac{(n-1)C_1(1 + r\sqrt{K}) + C_2}{r^2}$$

for some positive constants C_1, C_2. Because

$$(\partial_t - \Delta_V)(\varphi \chi G) = \varphi' \chi G + \varphi \chi (\partial_t - \Delta_V) G - \varphi G \Delta_V \chi - 2\varphi \left\langle \frac{\nabla(\varphi \chi G)}{\varphi \chi} - G \frac{\nabla \chi}{\chi}, \nabla \chi \right\rangle,$$

applying the above inequalities to $\varphi \chi G$ yields

$$\begin{aligned} (\partial_t - \Delta_V)(\varphi \chi G) &\leq \left[\varphi' \chi + 2\varphi \frac{C_1^2}{r^2} + \varphi \frac{(n-1)C_1(1 + r\sqrt{K}) + C_2}{r^2} \right] G \\ &\quad + \varphi \chi \left[-\frac{2(a-2)}{(a+1)^3} \frac{G^2}{A^4} + \frac{2(a+1)}{a} KG \right] - 2 \left\langle \nabla(\varphi \chi G), \frac{\nabla \chi}{\chi} \right\rangle \\ &= -\frac{2(a-2)}{(a+1)^3 A^4} \varphi \chi G^2 - 2 \left\langle \nabla(\varphi \chi G), \frac{\nabla \chi}{\chi} \right\rangle + \left[\left(\varphi' + \frac{2(a+1)}{a} K \varphi \right) \chi \right. \\ &\quad \left. + \frac{\varphi}{r^2} \left(2C_1^2 + C_2 + (n-1)C_1(1 + r\sqrt{K}) \right) \right] G. \end{aligned}$$

Let (x_0, t_0) be a point where $\varphi\chi G$ achieves its maximum. Then

$$\varphi\chi G \leq \frac{(a+1)^3A^4}{2(a-2)}\left[\varphi' + \frac{2(a+1)}{a}K\varphi + \frac{\varphi}{r^2}\left(C_3 + C_1(n-1)(1+r\sqrt{K})\right)\right]$$

at the point (x_0, t_0), where $C_3 := 2C_1^2 + C_2$. Locating on $B(x_0, r) \times (0, T]$ we derive the desired inequality. $\square$

Using (5.10) we obtain

$$\varphi|\nabla u_\epsilon|^2 \leq C\frac{1+r+r^2}{r^2}A^2$$

for some positive constant C depending only on n, K. Therefore

$$\int_0^T\int_{B(x_0,r)}\left(\frac{\varphi}{\epsilon}|\nabla u_\epsilon|^2\right)^2 e^f dVdt \leq \frac{C^2TA^4}{\epsilon^2}\frac{(1+r+r^2)^2}{r^4}\int_{B(x_0,r)} e^f dV =: e^{\alpha(r)}.$$

By the Bishop–Gromov volume comparison theorem for Δ_f (see [28], or [37, Theorem 4.1]), we see that $\int^\infty rdr/\alpha(r)$ is infinity and hence by Karp–Li–Grigor'yan's maximum principle we obtain $F_\epsilon \leq 0$. Letting $\epsilon \to 0$ implies (5.9). $\square$

Remark 5.12. We compare other Hamilton's estimates with (5.9). In our geometric proof we require the curvature condition $\mathrm{Ric}_f^{n,m} \geq -K$ in order to use the Bakry–Qian's Laplacian comparison theorem without any additional requirement on the potential function f. If we use the curvature condition $\mathrm{Ric}_f \geq -K$ in our geometric proof, then some conditions on f would be required (see [11,37]). A probabilistic proof of Li [24] shows a similar estimate

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{2}{t} + 2K\right)\ln\frac{A}{u}$$

where $0 < u \leq A$ on $\mathcal{M} \times (0, T]$ and $\mathrm{Ric}_f \geq -K$.

6. Hessian estimates

In this section we generalize Hessian estimates of the heat equation in [18] to the V-heat equation.

Theorem 6.1. *Let $(\mathcal{M}, g)$ be a closed m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -K$ where $K \geq 0$.*

(a) *If u is a solution of $\partial_t u = \Delta_V u$ in $\mathcal{M} \times (0, T]$ and $0 < u \leq A$, then*

$$\nabla^2 u \leq \left(B + \frac{5}{t}\right)u\left(1 + \ln\frac{A}{u}\right)g \tag{6.1}$$

in $\mathcal{M} \times (0, T]$, where

$$B = \sqrt{16m^{\frac{3}{2}}K_1\sup_{\mathcal{M}}|V|^2 + 2mK_2 + 3mKK_2 + 14m^{\frac{3}{2}}nKK_1 + 100n^2m^3(K_1+K_2)^2}$$

with $K_1 = \max_{\mathcal{M}}(|\mathrm{Rm}| + |\mathrm{Ric}_V|)$ and $K_2 = \max_{\mathcal{M}}|\nabla\mathrm{Ric}_V|$.

(b) *If u is a solution of $\partial_t u = \Delta_V u$ in $Q_{R,T}(x_0, t_0)$ and $0 < u \leq A$, then*

$$\nabla^2 u \leq C_1\left(\frac{1}{T} + \frac{1+R\sqrt{K}}{R^2} + B\right)u\left(1 + \ln\frac{A}{u}\right)^2 g \tag{6.2}$$

in $Q_{R/2,T/2}(x_0, t_0)$, where

$$B = C_2m^{5/2}n^2\left[K_1 + K_2 + \sqrt{(K_1+K_2)K + K_2 + K_1\sup_{\mathcal{M}}|V|^2}\right]$$

and C_1, C_2 are positive universal constants.

Let $(\mathcal{M}, g)$ be a closed m-dimensional Riemannian manifold with $\mathrm{Ric}_V^{n,m} \geq -K$ where $K \geq 0$, and u a solution of

$$\partial_t u = \Delta_V u \tag{6.3}$$

in $\mathcal{M} \times (0, T]$, where $T \in (0, \infty)$, and $0 < u \leq A$. Set

$$f := \ln\frac{u}{A} \tag{6.4}$$

as in [18]. Then

$$\nabla f=\frac{\nabla u}{u},\qquad \nabla^2 f=\frac{\nabla^2 u}{u}-\frac{\nabla u\otimes\nabla u}{u^2},\qquad \Delta f=\frac{\Delta u}{u}-\frac{|\nabla u|^2}{u^2},$$

and

$$\partial_t f=\frac{\partial_t u}{u}=\frac{\Delta_V u}{u}=\Delta_V f+|\nabla f|^2. \tag{6.5}$$

As in [18], we introduce the following quantities

$$v_{ij}:=\frac{\nabla_i\nabla_j u}{u(1-f)},\qquad w_{ij}:=\frac{\nabla_i u\nabla_j u}{u^2(1-f)^2}, \tag{6.6}$$

$$V:=(v_{ij}),\qquad W:=(w_{ij}),\qquad w:=\mathrm{tr}(W)=\frac{|\nabla u|^2}{u^2(1-f)^2}. \tag{6.7}$$

Using $\partial_t(u(1-f))=(1-f)\partial_t u-u\partial_t f=-f\partial_t u$ we have

$$\partial_t v_{ij}=\frac{\nabla_i\nabla_j\partial_t u}{u(1-f)}+f\frac{\partial_t u\nabla_i\nabla_j u}{u^2(1-f)^2}. \tag{6.8}$$

Similarly,

$$\nabla_k v_{ij}=\frac{\nabla_k\nabla_i\nabla_j u}{u(1-f)}+f\frac{\nabla_k u\nabla_i\nabla_j u}{u^2(1-f)^2}. \tag{6.9}$$

By the commutation formula (see [18, page 4]) we have

$$\begin{aligned}\partial_t\nabla_i\nabla_j u &= \nabla_i\nabla_j\left(\Delta u+\langle V,\nabla u\rangle\right)\\ &= \Delta\nabla_i\nabla_j u+2R_{kij\ell}\nabla^k\nabla^\ell u-R_{i\ell}\nabla_j\nabla^\ell u-R_{j\ell}\nabla_i\nabla^\ell u-\left(\nabla_i R_{j\ell}+\nabla_j R_{i\ell}-\nabla_\ell R_{ij}\right)\nabla^\ell u+\nabla_i\nabla_j\langle V,\nabla u\rangle.\end{aligned}$$

The last term on the right-hand side is equal to

$$\begin{aligned}\nabla_i\nabla_j\langle V,\nabla u\rangle &= \nabla_i\left(\nabla_k u\nabla_j V^k+V^k\nabla_j\nabla_k u\right)\\ &= \nabla_k u\nabla_i\nabla_j V^k+\nabla_i\nabla_k u\nabla_j V^k+\nabla_i V^k\nabla_j\nabla_k u+V^k\nabla_i\nabla_j\nabla_k u;\end{aligned}$$

using the commutation formula

$$\nabla_i\nabla_j\nabla_k u=\nabla_i\nabla_k\nabla_j u=\nabla_k\nabla_i\nabla_j u-R_{ikj\ell}\nabla^\ell u$$

we arrive at

$$\nabla_i\nabla_j\langle V,\nabla u\rangle=V^k\nabla_k\nabla_i\nabla_j u+R_{kij\ell}V^k\nabla^\ell u+\nabla_k u\nabla_i\nabla_j V^k+\nabla_i\nabla_k u\nabla_j V^k+\nabla_j\nabla_k u\nabla_i V^k.$$

Therefore

$$\begin{aligned}\partial_t\nabla_i\nabla_j u &= \Delta_V\nabla_i\nabla_j u+R_{kij\ell}\left(2\nabla^k\nabla^\ell u+V^k\nabla^\ell u\right)-\left(\nabla_i R_j{}^k+\nabla_j R_i{}^k-\nabla^k R_{ij}-\nabla_i\nabla_j V^k\right)\nabla_k u\\ &\quad -\nabla_i\nabla_k u\left(R_j{}^k-\nabla_j V^k\right)-\nabla_j\nabla_k u\left(R_i{}^k-\nabla_i V^k\right).\end{aligned} \tag{6.10}$$

Interchanging i and j in (6.10) and then adding it into (6.10) imply

$$\begin{aligned}\left(\partial_t-\Delta_V\right)\nabla_i\nabla_j u &= R_{kij\ell}\left(2\nabla^k\nabla^\ell u+\frac{V^k\nabla^\ell u+V^\ell\nabla^k u}{2}\right)-\left(\nabla_i R_j{}^k+\nabla_j R_i{}^k-\nabla^k R_{ij}-\frac{\nabla_i\nabla_j V^k+\nabla_j\nabla_i V^k}{2}\right)\nabla_k u\\ &\quad -\nabla_i\nabla_k u\left(R_j{}^k-\nabla_j V^k\right)-\nabla_j\nabla_k u\left(R_i{}^k-\nabla_i V^k\right).\end{aligned} \tag{6.11}$$

Recall the Bakry–Emery–Ricci curvatures

$$\mathrm{Ric}_V:=\mathrm{Ric}-\frac{1}{2}\mathscr{L}_V g,\qquad \mathrm{Ric}_V^{n,m}:=\mathrm{Ric}_V-\frac{1}{n-m}V\otimes V.$$

Then

$$\begin{aligned}\nabla_k u\nabla_i(\mathrm{Ric}_V)_j{}^k &= \nabla_k u\nabla_i\left(R_j{}^k-\frac{\nabla_j V^k+\nabla^k V_j}{2}\right)\\ &= \nabla_k u\nabla_i\left(R_j{}^k-\frac{1}{2}\nabla_j V^k\right)-\frac{1}{2}\nabla_k u\nabla_i\nabla^k V_j,\end{aligned}$$

$$\begin{aligned}\nabla_k u \nabla^k (\mathrm{Ric}_V)_{ij} &= \nabla_k u \nabla^k \left(R_{ij} - \frac{\nabla_i V_j + \nabla_j V_i}{2} \right) \\ &= \nabla_k u \nabla^k R_{ij} - \frac{1}{2} \nabla^k u \left(\nabla_k \nabla_i V_j + \nabla_k \nabla_j V_i \right) \\ &= \nabla_k u \nabla^k R_{ij} - \frac{1}{2} \nabla^k u \left(\nabla_i \nabla_k V_j + \nabla_j \nabla_k V_i - R_{kij\ell} V^\ell - R_{kji\ell} V^\ell \right).\end{aligned}$$

The middle term on the right-hand side of (6.11) can be now rewritten as

$$\begin{aligned}&\left(\nabla_i R_j{}^k + \nabla_j R_i{}^k - \nabla^k R_{ij} - \frac{\nabla_i \nabla_j V^k + \nabla_j \nabla_i V^k}{2} \right) \nabla_k u \\ &\quad = \left[\nabla_i (\mathrm{Ric}_V)_j{}^k + \nabla_j (\mathrm{Ric}_V)_i{}^k - \nabla^k (\mathrm{Ric}_V)_{ij} \right] \nabla_k u + \frac{1}{2} R_{kij\ell} \left(V^\ell \nabla^k u + V^k \nabla^\ell u \right).\end{aligned}$$

Therefore

$$\begin{aligned}(\partial_t - \Delta_V)\, \nabla_i \nabla_j u &= 2R_{kij\ell} \nabla^k \nabla^\ell u - \nabla_i \nabla_k u (\mathrm{Ric}_V)_j{}^k - \nabla_j \nabla_k u (\mathrm{Ric}_V)_i{}^k - \left(\nabla_i (\mathrm{Ric}_V)_j{}^k + \nabla_j (\mathrm{Ric}_V)_i{}^k - \nabla^k (\mathrm{Ric}_V)_{ij} \right) \nabla_k u \\ &\quad - \nabla_i \nabla^k u \frac{\nabla_k V_j - \nabla_j V_k}{2} - \nabla_j \nabla^k u \frac{\nabla_k V_i - \nabla_i V_k}{2}. \qquad (6.12)\end{aligned}$$

Lemma 6.2. *We have*

$$\begin{aligned}(\partial_t - \Delta_V)\, v_{ij} &= -\frac{2f}{1-f} \nabla^k f \nabla_k v_{ij} - \frac{|\nabla f|^2}{1-f} v_{ij} + \frac{1}{u(1-f)} \Big[2R_{kij\ell} \nabla^k \nabla^\ell u \\ &\quad - \nabla_i \nabla_k u (\mathrm{Ric}_V)_j{}^k - \nabla_j \nabla_k (\mathrm{Ric}_V)_i{}^k - \Big(\nabla_i (\mathrm{Ric}_V)_j{}^k \\ &\quad + \nabla_j (\mathrm{Ric}_V)_i{}^k - \nabla^k (\mathrm{Ric}_V)_{ij} \Big) \nabla_k u - \nabla_i \nabla^k u \frac{\nabla_k V_j - \nabla_j V_k}{2} - \nabla_j \nabla^k u \frac{\nabla_k V_i - \nabla_i V_k}{2} \Big].\end{aligned}$$

Proof. Using $u \nabla f = \nabla u$ and an identity in [18, page 4, line -6] we have

$$\begin{aligned}\Delta_V v_{ij} &= \Delta v_{ij} + V^k \nabla_k v_{ij} \\ &= \frac{\Delta \nabla_i \nabla_j u}{u(1-f)} + \frac{f \Delta u \nabla_i \nabla_j u}{u^2(1-f)^2} + \frac{2f \nabla^k u \nabla_k \nabla_i \nabla_j u}{u^2(1-f)^2} + \frac{\nabla_i \nabla_j u \langle \nabla u, \nabla f \rangle}{u^2(1-f)^2} \\ &\quad + \frac{2f^2 \nabla_i \nabla_j u |\nabla u|^2}{u^3(1-f)^3} + \frac{V^k \nabla_k \nabla_i \nabla_j u}{u(1-f)} + \frac{\langle V, \nabla u \rangle f \nabla_i \nabla_j u}{u^2(1-f)^2} \\ &= \frac{\Delta_V \nabla_i \nabla_j u}{u(1-f)} + \frac{f \Delta_V u \nabla_i \nabla_j u}{u^2(1-f)^2} + \frac{2f \nabla^k u \nabla_k \nabla_i \nabla_j u}{u^2(1-f)^2} + \frac{\nabla_i \nabla_j u \langle \nabla u, \nabla f \rangle}{u^2(1-f)^2} + \frac{2f^2 \nabla_i \nabla_j u |\nabla u|^2}{u^3(1-f)^3}.\end{aligned}$$

Similarly,

$$\begin{aligned}\partial_t v_{ij} &= \frac{\partial_t \nabla_i \nabla_j u}{u(1-f)} - \frac{\nabla_i \nabla_j u}{u^2(1-f)^2} [\partial_t u (1-f) - u \partial_t f] \\ &= \frac{\partial_t \nabla_i \nabla_j u}{u(1-f)} + \frac{f \Delta_V u \nabla_i \nabla_j u}{u^2(1-f)^2}.\end{aligned}$$

Combining these two identities yields

$$(\partial_t - \Delta_V)\, v_{ij} = \frac{1}{u(1-f)} (\partial_t - \Delta_V)\, \nabla_i \nabla_j u - \frac{2f \nabla^k f \nabla_k \nabla_i \nabla_j u}{u(1-f)^2} - \frac{\nabla_i \nabla_j u |\nabla f|^2}{u(1-f)^2} - \frac{2\nabla_i \nabla_j u}{u(1-f)^3} f^2 |\nabla f|^2.$$

Using (6.9) and (6.12) we prove the desired identity. □

When V is gradient (i.e., $V = \nabla \phi$ for some smooth function ϕ on $\mathscr{M}$), Lemma 6.2 reduces to Lemma 2.1 in [18] where Δ is replaced by Δ_ϕ.

Lemma 6.3. *We have*

$$\begin{aligned}(\partial_t - \Delta_V)\, w_{ij} &= -\frac{2f}{1-f} \nabla^k f \nabla_k w_{ij} - \frac{2|\nabla f|^2}{1-f} w_{ij} - 2(v_{ik} + f w_{ik})(v_j{}^k + f w_j{}^k) - (\mathrm{Ric}_V)_i{}^k w_{jk} - (\mathrm{Ric}_V)_j{}^k w_{ik} \\ &\quad - w_i{}^k \frac{\nabla_k V_j - \nabla_j V_k}{2} - w_j{}^k \frac{\nabla_k V_i - \nabla_i V_k}{2}.\end{aligned}$$

Proof. Compute

$$\partial_t w_{ij} = \frac{\nabla_i u \nabla_j \partial_t u + \nabla_j u \nabla_i \partial_t u}{u^2(1-f)^2} + \frac{2f \partial_t u \nabla_i u \nabla_j u}{u^3(1-f)^3},$$
$$\nabla_k w_{ij} = \frac{\nabla_i u \nabla_j \nabla_k u + \nabla_j u \nabla_i \nabla_k u}{u^2(1-f)^2} + \frac{2f \nabla_i u \nabla_j u \nabla_k u}{u^3(1-f)^3}.$$

By the identity in [18, page 5, line 14], we have

$$\begin{aligned} \Delta_V w_{ij} &= \Delta w_{ij} + V^k \left(\frac{\nabla_i u \nabla_j \nabla_k u}{u^2(1-f)^2} + \frac{2f \nabla_i \nabla_j u \nabla_k u}{u^3(1-f)^3} \right) \\ &= \frac{\nabla_i u \nabla_j \Delta u + 2\nabla_i \nabla_k u \nabla_j \nabla^k u + \nabla_j u \nabla_i \Delta u}{u^2(1-f)^2} + R_i{}^k \frac{\nabla_k u \nabla_j u}{u^2(1-f)^2} \\ &\quad + R_j{}^k \frac{\nabla_k u \nabla_i u}{u^2(1-f)^2} + \frac{4f \nabla^k u (\nabla_i u \nabla_j \nabla_k u + \nabla_j u \nabla_i \nabla_k u)}{u^3(1-f)^3} \\ &\quad + \frac{2\nabla_i u \nabla_j u (\langle \nabla u, \nabla f \rangle + f \Delta u)}{u^3(1-f)^3} + \frac{6f^2 |\nabla u|^2 \nabla_i u \nabla_j u}{u^4(1-f)^4} \\ &\quad + \frac{V^k \nabla_k \nabla_j u \nabla_i u}{u^2(1-f)^2} + \frac{V^k \nabla_k \nabla_i u \nabla_j u}{u^2(1-f)^2} + \frac{2f \langle V, \nabla u \rangle \nabla_i u \nabla_j u}{u^3(1-f)^3}. \end{aligned}$$

Since $\Delta u = \Delta_V u - V^k \nabla_k u$, it follows that

$$\nabla_j \Delta u = \nabla_j \Delta_V u - \nabla_k u \nabla_j V^k - V^k \nabla_k \nabla_j u$$

and then

$$\nabla_i u \nabla_j \Delta u + V^k \nabla_k \nabla_j u \nabla_i u = \nabla_i u \nabla_j \Delta_V u - \nabla_i u \nabla_k u \nabla_j V^k.$$

On the other hand, we have

$$R_j{}^k \frac{\nabla_k u \nabla_i u}{u^2(1-f)^2} = (\mathrm{Ric}_V)_j{}^k \frac{\nabla_k u \nabla_i u}{u^2(1-f)^2} + \frac{\nabla_i u \nabla^k u (\nabla_j V_k + \nabla_k V_j)}{2u^2(1-f)^2}.$$

Similarly, we can find an analogue identity for $R_i{}^k \nabla_k u \nabla_j u / u^2(1-f)^2$. Therefore

$$\begin{aligned} \Delta_V w_{ij} &= \frac{\nabla_i u [\nabla_j \Delta_V u + (\mathrm{Ric}_V)_j{}^k \nabla_k u]}{u^2(1-f)^2} + \frac{\nabla_j u [\nabla_i \Delta_V u + (\mathrm{Ric}_V)_i{}^k \nabla_k u]}{u^2(1-f)^2} \\ &\quad + \frac{2\nabla_i \nabla_k u \nabla_j \nabla^k u}{u^2(1-f)^2} + \frac{4f \nabla^k u (\nabla_i u \nabla_j \nabla_k u + \nabla_j u \nabla_i \nabla_k u)}{u^3(1-f)^3} \\ &\quad + \frac{2\nabla_i u \nabla_j u (\langle \nabla u, \nabla f \rangle + f \Delta_V u)}{u^3(1-f)^3} + \frac{6f^2 |\nabla u|^2 \nabla_i u \nabla_j u}{u^4(1-f)^4} \\ &\quad + \frac{\nabla_i u \nabla^k u}{u^2(1-f)^2} \frac{\nabla_k V_j - \nabla_j V_k}{2} + \frac{\nabla_j u \nabla^k u}{u^2(1-f)^2} \frac{\nabla_k V_i - \nabla_i V_k}{2}. \end{aligned}$$

Together with the expression of $\partial_t w_{ij}$, we arrive at

$$\begin{aligned} (\partial_t - \Delta_V)\, w_{ij} &= -(\mathrm{Ric}_V)_i{}^k \frac{\nabla_k u \nabla_j u}{u^2(1-f)^2} - (\mathrm{Ric}_V)_j{}^k \frac{\nabla_k u \nabla_i u}{u^2(1-f)^2} \\ &\quad - \frac{2\nabla_i \nabla_k u \nabla_j \nabla^k u}{u^2(1-f)^2} - \frac{4f \nabla^k u (\nabla_i u \nabla_j \nabla_k u + \nabla_j u \nabla_i \nabla_k u)}{u^3(1-f)^3} \\ &\quad - \frac{2\nabla_i u \nabla_j u \langle \nabla u, \nabla f \rangle}{u^3(1-f)^3} - \frac{6f^2 |\nabla u|^2 \nabla_i u \nabla_j u}{u^4(1-f)^4} \\ &\quad - \frac{\nabla_i u \nabla^k u}{u^2(1-f)^2} \frac{\nabla_k V_j - \nabla_j V_k}{2} - \frac{\nabla_j u \nabla^k u}{u^2(1-f)^2} \frac{\nabla_k V_i - \nabla_i V_k}{2}. \end{aligned}$$

As in [18], the middle four terms H on the right-hand side can be written as

$$H = -\frac{2\nabla_i\nabla_k u\nabla_j\nabla^k u}{u^2(1-f)^2} - \frac{4f\nabla^k u(\nabla_i u\nabla_j\nabla_k u + \nabla_j u\nabla_i\nabla_k u)}{u^3(1-f)^3} - \frac{2\nabla_i u\nabla_j u\langle\nabla u, \nabla f\rangle}{u^3(1-f)^3} - \frac{6f^2|\nabla u|^2\nabla_i u\nabla_j u}{u^4(1-f)^4}$$
$$= -\frac{2f}{1-f}\nabla^k f\nabla_k w_{ij} - \frac{2|\nabla f|^2}{1-f}w_{ij} - 2(v_{ik}+fw_{ik})(v_j{}^k + fw_j{}^k).$$

Plugging the expression of H into $(\partial_t - \Delta_V)w_{ij}$ we obtain the result. $\square$

From (6.7) we see that

$$w = \frac{|\nabla f|^2}{(1-f)^2}$$

so that Lemmas 6.2 and 6.3 can be rewritten as

$$\begin{aligned}(\partial_t - \Delta_V)\, v_{ij} \;&=\; -\frac{2f}{1-f}\nabla^k f\nabla_k v_{ij} - (1-f)wv_{ij} + 2R_{kij\ell}v^{k\ell} - (\mathrm{Riv}_V)_{ik}v_j{}^k \\ &\quad - (\mathrm{Ric}_V)_{jk}v_i{}^k + v_i{}^k(\mathscr{A}_V g)_{jk} + v_j{}^k(\mathscr{A}_V g)_{ik} - \frac{\nabla^k u}{u(1-f)}\Big(\nabla_i(\mathrm{Ric}_V)_{jk} + \nabla_j(\mathrm{Ric}_V)_{ik} - \nabla_k(\mathrm{Ric}_V)_{ij}\Big),\\ (\partial_t - \Delta_V)\, w_{ij} \;&=\; -\frac{2f}{1-f}\nabla^k f\nabla_k w_{ij} - 2(1-f)ww_{ij} - 2(v_{ik}+fw_{ik})(v_j{}^k + fw_j{}^k)\\ &\quad - (\mathrm{Ric}_V)_{ik}w_j{}^k - (\mathrm{Ric}_V)_{jk}w_i{}^k + w_i{}^k(\mathscr{A}_V g)_{jk} + w_j{}^k(\mathscr{A}_V g)_{ik},\end{aligned}$$

where $\mathscr{A}_V g$ stands for the tensor field given by

$$(\mathscr{A}_V g)_{ij} := \frac{\nabla_i V_j - \nabla_j V_i}{2}. \tag{6.13}$$

The tensor field exactly the 2-form $dV_\flat$ where $V_\flat$ is the corresponding 1-form of V. When V is a gradient vector field $V = \nabla\phi$, we see that $\mathscr{A}_V g$ vanishes identically on $\mathscr{M}$. In this sense $\mathscr{A}_V g$ is an obstruction of V being gradient.

Let $p \in \mathscr{M}$ and choose a local orthonormal coordinates $(x^i)_{1\le i\le m}$ around p. We follow the method in [18]. Consider the operator

$$\Box_V := \partial_t - \Delta_V + \frac{2f}{1-f}\langle\nabla f, \nabla\rangle. \tag{6.14}$$

Then the matrices $\boldsymbol{V} = (v_{ij})$ and $\boldsymbol{W} = (w_{ij})$ satisfy

$$\Box_V \boldsymbol{V} = -(1-f)w\boldsymbol{V} - \boldsymbol{P} - \boldsymbol{VA} + \boldsymbol{AV}, \tag{6.15}$$

$$\Box_V \boldsymbol{W} = -2(1-f)w\boldsymbol{W} - 2(\boldsymbol{V}+f\boldsymbol{W})^2 - \boldsymbol{Q} - \boldsymbol{WA} + \boldsymbol{AW}, \tag{6.16}$$

where $\boldsymbol{P} = (P_{ij})$, $\boldsymbol{Q} = (Q_{ij})$, $\boldsymbol{A} = (A_{ij})$ are matrices whose entries are

$$\begin{aligned}P_{ij} \;:=\; &-2\sum_{1\le k,\ell\le m} R_{kij\ell}v_{k\ell} + \sum_{1\le k\le m}\Bigg[(\mathrm{Ric}_V)_{ik}v_{kj} + v_{ik}(\mathrm{Ric}_V)_{kj}\\ &+ \frac{\nabla_k u}{u(1-f)}\Big(\nabla_i(\mathrm{Ric}_V)_{jk} + \nabla_j(\mathrm{Ric}_V)_{ik} - \nabla_k(\mathrm{Ric}_V)_{ij}\Big)\Bigg],\end{aligned} \tag{6.17}$$

$$Q_{ij} := \sum_{1\le k\le m}\Big((\mathrm{Ric}_V)_{ik}w_{kj} + w_{ik}(\mathrm{Ric}_V)_{kj}\Big), \tag{6.18}$$

$$A_{ij} := (\mathscr{A}_V g)_{ij}. \tag{6.19}$$

For any real number α we define

$$\boldsymbol{V} \oplus_\alpha \boldsymbol{W} := \alpha\boldsymbol{V} + \boldsymbol{W}. \tag{6.20}$$

Then

$$\Box_V(\boldsymbol{V}\oplus_\alpha\boldsymbol{W}) = -\alpha(1-f)w\boldsymbol{V} - 2(1-f)w\boldsymbol{V} - 2(\boldsymbol{V}+f\boldsymbol{W})^2 - \boldsymbol{P}\oplus_\alpha\boldsymbol{Q} - (\boldsymbol{V}\oplus_\alpha\boldsymbol{W})\boldsymbol{A} + \boldsymbol{A}(\boldsymbol{V}\oplus_\alpha\boldsymbol{W}). \tag{6.21}$$

Let $\boldsymbol{\xi} \in T_p\mathscr{M} \cong \mathbf{R}^m$ be a unit eigenvector of $\boldsymbol{V}\oplus_\alpha\boldsymbol{W}$, i.e., $(\boldsymbol{V}\oplus_\alpha\boldsymbol{W})\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$. By parallel translation along geodesics, we extend $\boldsymbol{\xi}$ to a smooth vector field, still denoted by $\boldsymbol{\xi}$, near p. Then

$$\lambda = (\boldsymbol{V}\oplus_\alpha\boldsymbol{W})(\boldsymbol{\xi}, \boldsymbol{\xi}) \tag{6.22}$$

is a smooth function near p. From (6.21) and (6.22) we obtain

$$\begin{aligned}\Box_V\lambda &= -\alpha(1-f)w\boldsymbol{V}(\boldsymbol{\xi},\boldsymbol{\xi}) - 2(1-f)w\boldsymbol{W}(\boldsymbol{\xi},\boldsymbol{\xi}) - 2|(\boldsymbol{V}+f\boldsymbol{W})\boldsymbol{\xi}|^2 \\ &\quad - (\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi}) - ((\boldsymbol{V}\oplus_\alpha\boldsymbol{W})\boldsymbol{A})(\boldsymbol{\xi},\boldsymbol{\xi}) + (\boldsymbol{A}(\boldsymbol{V}\oplus_\alpha\boldsymbol{W}))(\boldsymbol{\xi},\boldsymbol{\xi}) \\ &\le -\frac{2\lambda^2}{\alpha^2} - \lambda\left(w - \frac{4}{\alpha^2}\boldsymbol{W}(\boldsymbol{\xi},\boldsymbol{\xi})\right) + f\lambda\left(w - \frac{4}{\alpha}\boldsymbol{W}(\boldsymbol{\xi},\boldsymbol{\xi})\right) - (\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})\end{aligned}$$

where we used the estimate (2.6) in [18] and

$$((\boldsymbol{V}\oplus_\alpha\boldsymbol{W})\boldsymbol{A})(\boldsymbol{\xi},\boldsymbol{\xi}) = \lambda\boldsymbol{A}(\boldsymbol{\xi},\boldsymbol{\xi}) = (\boldsymbol{A}(\boldsymbol{V}\oplus_\alpha\boldsymbol{W}))(\boldsymbol{\xi},\boldsymbol{\xi}).$$

Since $\boldsymbol{W}(\xi,\xi)\le w$, it follows from (2.7) in [18] that

$$\Box_V\lambda \le -\frac{2\lambda^2}{\alpha^2} - (\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi}) \quad \text{at } p, \ \text{whenever } \lambda\ge 0, \tag{6.23}$$

where $\alpha\ge 4$.

Proof part (a) of Theorem 6.1. As in [18], we consider the quantity

$$\boldsymbol{V}\oplus_{\alpha,\tau}\boldsymbol{W} := \alpha\boldsymbol{V} + \boldsymbol{W} - \frac{\tau}{t}\boldsymbol{g} \tag{6.24}$$

where $\boldsymbol{g} := (g_{ij})$ and τ is a positive constant determined later. Assume now that $\boldsymbol{V}\oplus_{\alpha,\tau}\boldsymbol{W}$ has the largest nonnegative eigenvalue with the unit eigenvector $\boldsymbol{\xi}$ at a point (p_1,t_1) with $t_1>0$. As before we consider

$$\lambda := (\boldsymbol{V}\oplus_\alpha\boldsymbol{W})(\boldsymbol{\xi},\boldsymbol{\xi}), \qquad \mu := (\boldsymbol{V}\oplus_{\alpha,\tau}\boldsymbol{W})(\boldsymbol{\xi},\boldsymbol{\xi}) = \lambda - \frac{\tau}{t}.$$

Since μ has its nonnegative maximum at (p_1,t_1), it follows that $\Delta\mu\le 0=\nabla\mu\le\partial_t\mu$ and hence $\Box_V\mu\le 0$ at (p_1,t_1). Consequently,

$$\frac{2\lambda^2}{\alpha^2} \le \frac{\tau}{t^2} + |(\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})| \quad \text{at } (p_1,t_1) \tag{6.25}$$

as that of (2.11) in [18]. Let $\boldsymbol{\xi}=(\xi_1,\dots,\xi^m)^T$ and note that

$$\begin{aligned}|(\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})| &\le \alpha|\boldsymbol{P}(\boldsymbol{\xi},\boldsymbol{\xi})| + |\boldsymbol{Q}(\boldsymbol{\xi},\boldsymbol{\xi})| \\ &\le \alpha\left|\sum_{1\le i,j\le m}\xi_i\xi_j\left(-2\sum_{1\le k,\ell\le m}R_{kij\ell}v_{k\ell} + \sum_{1\le k\le m}R^V_{ik}v_{kj} + \sum_{1\le k\le m}v_{ik}R^V_{kj}\right)\right| \\ &\quad + \left|\sum_{1\le i,j,k\le m}\xi_i\xi_j\left(R^V_{ik}w_{kj} + w_{ik}R^V_{kj}\right)\right| + \alpha\left|\sum_{1\le i,j,k\le m}\xi_i\xi_j\frac{\nabla_k u}{u(1-f)}\left(\nabla_i R^V_{jk} + \nabla_j R^V_{ik} - \nabla_k R^V_{ij}\right)\right|\end{aligned}$$

where $R^V_{ij} := (\mathrm{Ric}_V)_{ij}$. Since $\boldsymbol{\xi}$ is unit, it follows that

$$\begin{aligned}\left|\sum_{1\le i,j,k\le m}\xi_i\xi_j\left(R^V_{ik}w_{kj} + w_{ik}R^V_{kj}\right)\right| &\le \sum_{1\le i,j,k\le m}\left|R^V_{ik}w_{kj} + w_{ik}R^V_{kj}\right| \\ &\le 2\left(\sum_{1\le i,j,k\le m}(R^V_{ik})^2\right)^{\frac12}\left(\sum_{1\le i,j,k\le m}w^2_{kj}\right)^{\frac12} \\ &\le 2m|\mathrm{Ric}_V|\,|\boldsymbol{W}|.\end{aligned}$$

Similarly,

$$\left|\sum_{1\le i,j,k\le m}\xi_i\xi_j\frac{\nabla_k u}{u(1-f)}\left(\nabla_i R^V_{jk} + \nabla_j R^V_{ik} - \nabla_k R^V_{ij}\right)\right| \le 3m|\nabla\mathrm{Ric}_V|\,|\boldsymbol{W}|^{1/2}.$$

As the inequality (2.12) in [18], we have

$$\begin{aligned}|(\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})| \le &\left|\sum_{1\le i,j\le m}\xi_i\xi_j\left(-2\sum_{1\le k,\le m}R_{kij\ell}(\alpha v_{k\ell} + w_{k\ell})\right.\right. \\ &\left.\left.+ \sum_{1\le k\le m}R^V_{ik}(\alpha v_{kj} + w_{kj}) + \sum_{1\le k\le m}(\alpha v_{ik} + w_{ik})R^V_{kj}\right)\right|\end{aligned}$$

$$+\left|\sum_{1\le i,j\le m}\xi_i\xi_j\left(-2\sum_{1\le k,\ell\le m}R_{kij\ell}w_{k\ell}+\sum_{1\le k\le m}R^V_{ik}w_{kj}\right.\right.$$
$$\left.\left.+\sum_{1\le k\le m}w_{ik}R^V_{kj}\right)\right|+3m|\nabla\mathrm{Ric}_V|\,|\boldsymbol{W}|^{1/2}+2m|\mathrm{Ric}_V|\,|\boldsymbol{W}|. \tag{6.26}$$

In order to bound the function $|(\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})|$ at the point p_1, as in [18], we choose a local coordinate system so that the matrix $\boldsymbol{V}\oplus_\alpha\boldsymbol{W}$ is diagonal and $\boldsymbol{V}\oplus_\alpha\boldsymbol{W}-\frac{\tau}{t}\boldsymbol{g}=\mathrm{diag}(\mu_1,\dots,\mu_m)$ with $\mu_1\le\cdots\le\mu_m$ and $\mu_1<0<\mu_m$. Then

$$\begin{aligned}\left|\sum_{1\le i,j,k,\ell\le m}\xi_i\xi_jR_{kij\ell}(\alpha v_{k\ell}+w_{k\ell})\right| &\le \sum_{1\le i,j,k,\ell\le m}\left|R_{kij\ell}\left(\alpha v_{k\ell}+w_{k\ell}-\frac{\tau}{t}g_{k\ell}\right)\right|+\sum_{1\le i,j,k,\ell\le m}\left|R_{kij\ell}g_{k\ell}\right|\frac{\tau}{t}\\ &=\sum_{1\le i,j,k\le m}\left|R_{kijk}\mu_k\right|+\sum_{1\le i,j,k\le m}\left|R_{kijk}\right|\frac{\tau}{t}\\ &\le\left(\sum_{1\le i,j,k\le m}R^2_{kijk}\right)^{1/2}\left[\left(\sum_{1\le i,j,k\le m}\mu_k^2\right)^{1/2}+\left(\sum_{1\le i,j,k\le m}1\right)^{1/2}\frac{\tau}{t}\right]\\ &\le|\mathrm{Rm}|\left(m\left(\sum_{1\le k\le m}\mu_k^2\right)^{1/2}+m^{3/2}\frac{\tau}{t}\right)\\ &\le|\mathrm{Rm}|\left(m^{3/2}(\mu_m+|\mu_1|)+m^{3/2}\frac{\tau}{t}\right)=m^{3/2}|\mathrm{Rm}|\left(\mu_m+|\mu_1|+\frac{\tau}{t}\right).\end{aligned}$$

Here we used the estimate that

$$\left(\sum_{1\le k\le m}\mu_k^2\right)^{\frac12}\le\left((m-i)\mu_m^2+i\mu_1^2\right)^{\frac12}\le\left(\sqrt{m-i}\mu_m+\sqrt{i}|\mu_1|\right)^{\frac12}\le\sqrt{m}(\mu_m+|\mu_1|)$$

where μ_i is the largest eigenvalue so that $\mu_i<0$ but $\mu_{i+1}\ge0$. Similarly, we have

$$\begin{aligned}\left|\sum_{1\le i,j,k\le m}\xi_i\xi_jR^V_{ik}(\alpha v_{kj}+w_{kj})\right| &\le\sum_{1\le i,j,k\le m}\left|R^V_{ik}\left(\alpha v_{kj}+w_{kj}-\frac{\tau}{t}g_{kj}\right)\right|+\sum_{1\le i,j,k\le m}|R^V_{ik}g_{kj}|\frac{\tau}{t}\\ &=\sum_{1\le i,j\le m}|R^V_{ij}\mu_j|+\sum_{1\le i,j\le m}|R^V_{ij}|\frac{\tau}{t}\\ &\le\left(\sum_{1\le i,j\le m}|R^V_{ij}|^2\right)^{1/2}\left[\left(\sum_{1\le i,j\le m}\mu_j^2\right)^{1/2}+m\frac{\tau}{t}\right]\\ &\le m|\mathrm{Ric}_V|\left[\sqrt{m}(\mu_m+|\mu_1|)+\frac{\tau}{t}\right]\le m^{3/2}|\mathrm{Ric}_V|\left(\mu_m+|\mu_1|+\frac{\tau}{t}\right).\end{aligned}$$

Plugging those estimates into (6.26) yields

$$\begin{aligned}&|(\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})|\\ &\quad\le2m^{3/2}\Big(|\mathrm{Rm}|+|\mathrm{Ric}_V|\Big)\left(\mu_m+|\mu_1|+\frac{\tau}{t}\right)+3m|\nabla\mathrm{Ric}_V|\,|\boldsymbol{W}|^{1/2}+4m\Big(|\mathrm{Rm}|+|\mathrm{Ric}_V|\Big)|\boldsymbol{W}|.\end{aligned}\tag{6.27}$$

Set

$$K_1:=\max_{\mathcal{M}}\Big(|\mathrm{Rm}|+|\mathrm{Ric}_V|\Big),\qquad K_2:=\max_{\mathcal{M}}|\nabla\mathrm{Ric}_V|.\tag{6.28}$$

Therefore, using $2|\boldsymbol{W}|^{1/2}\le1+|\boldsymbol{W}|$, we arrive at

$$|(\boldsymbol{P}\oplus_\alpha\boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})|\le2m^{3/2}K_1\left(\mu_m+|\mu_1|+\frac{\tau}{t}\right)+2mK_2+4m(K_1+K_2)|\boldsymbol{W}|.\tag{6.29}$$

By the page 9 in [18], we have

$$\mu_m+|\mu_1|\le m\mu_m-\frac{\alpha\Delta u}{u(1-f)}-\frac{|\nabla u|^2}{u^2(1-f)^2}+\frac{m\tau}{t}.$$

By (5.5), we deduce that

$$-\frac{\alpha\Delta u}{u} \le \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\alpha-1} + \frac{\alpha}{u}\langle V, \nabla u\rangle - \frac{|\nabla u|^2}{u^2} \le \frac{n\alpha^2}{2t} + \frac{n\alpha^2 K}{\alpha-1} + \frac{\alpha^2}{2}|V|^2 - \frac{|\nabla u|^2}{2u^2}.$$

Since $1/(1-f) \le 1$ it follows that

$$\begin{aligned}|(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi})| \le\ & 2m^{3/2}K_1\left(m\mu_m + \frac{n\alpha^2 + 2\tau}{2t}\right) + 4m(K_1+K_2)|\boldsymbol{W}| \\ & + 2m^{3/2}K_1\left(\frac{n\alpha^2 K}{\alpha-1} + \frac{\alpha^2}{2}|V|^2\right) + 2mK_2 \end{aligned} \tag{6.30}$$

at the point (p_1, t_1). Because $\mathrm{Riv}_V^{n,m} \ge -K$ implies $\mathrm{Ric}_V \ge -K$, the estimate (5.7) tells us that

$$|\boldsymbol{W}| = \frac{|\nabla u|^2}{u^2(1-f)^2} \le \left(\frac{1}{t} + 2K\right)\frac{-f}{(1-f)^2} \le \frac{1}{4}\left(\frac{1}{t} + 2K\right).$$

Since $\mu = \mu_m < \lambda$ at (p_1, t_1), by the same argument in the page 10 of [18], we obtain

$$\begin{aligned}\frac{2\lambda^2}{\alpha^2} \le\ & \frac{\tau}{t^2} + m^{3/2}(K_1+K_2)\left(2m\lambda + \frac{n\alpha^2 + 2\tau + 1}{t}\right) + m^{3/2}\alpha^2|V|^2K_1 \\ & + 2mK_2 + mKK_2 + \left(\frac{2m^{3/2}n\alpha^2}{\alpha-1} + m\right)KK_1 \end{aligned} \tag{6.31}$$

from (6.25), at the point (p_1, t_1). By assumption $n \ge m$ and $\alpha \ge 4$, we have

$$n\alpha^2 + 2\tau + 1 \le n\alpha^2 + \alpha^2\tau + \alpha^2 \le (n+1)\alpha^2(1+\tau)$$

and hence

$$m^{3/2}(K_1+K_2)\left(2m\lambda + \frac{n\alpha^2 + 2\tau + 1}{t}\right) \le 2nm^{3/2}\alpha^2(K_1+K_2)\left(\lambda + \frac{1+\tau}{t}\right).$$

Letting

$$\begin{aligned} B_1 &:= 2nm^{3/2}\alpha^2(K_1+K_2), \\ B_2 &:= m^{3/2}\alpha^2|V|^2K_1 + 2mK_2 + mKK_2 + \left(\frac{2m^{3/2}n\alpha^2}{\alpha-1} + m\right)KK_1 \end{aligned}$$

we conclude from (6.31) that

$$\frac{2\lambda^2}{\alpha^2} \le \frac{\tau}{t^2} + B_1\left(\alpha\frac{\lambda}{\alpha} + \frac{1+\tau}{t}\right) + B_2. \tag{6.32}$$

By Cauchy's inequality, we get $B_1\lambda \le \frac{\lambda^2}{\alpha^2} + \frac{\alpha^2 B_1^2}{4}$ and hence

$$\frac{\lambda^2}{\alpha^2} \le \frac{\tau+1}{t^2} + \frac{B_1\sqrt{\tau+1}}{2}2\frac{\sqrt{\tau+1}}{t} + B_2 + \frac{\alpha^2 B_1^2}{4}.$$

Putting

$$B := \max\left\{\frac{B_1\sqrt{\tau+1}}{2}, \sqrt{B_2 + \frac{1}{4}\alpha^2 B_1^2}\right\}$$

the above inequality yields

$$\frac{\lambda}{\alpha} \le \frac{\sqrt{\tau+1}}{t} + B \tag{6.33}$$

at the point (p_1, t_1). As in the page 10 of [18], we then arrive at

$$(\boldsymbol{V} \oplus_\alpha \boldsymbol{W})(\boldsymbol{\eta}, \boldsymbol{\eta}) - \frac{\tau}{t} \le \left(\lambda - \frac{\tau}{t}\right)_{(p_1,t_1)} \le \frac{\alpha\sqrt{\tau+1} - \tau}{t} + \alpha B$$

in $\mathcal{M} \times (0, T]$. If we choose $\alpha := \frac{\tau}{\sqrt{\tau+1}} \geq 4$, then

$$t|\nabla^2 u| \leq \left(\sqrt{\tau+1} + Bt\right) u \left(1 - \ln\frac{u}{A}\right)$$

where $0 < u \leq A$ and $\tau \geq 4\sqrt{\tau+1}$. The restriction on τ implies that $\tau \geq 8 + 4\sqrt{5}$ and that we can take $\tau := 8 + 4\sqrt{5}$ and then $\alpha = 4$. Hence

$$t|\nabla^2 u| \leq \left(2 + \sqrt{5} + Bt\right) u \left(1 - \ln\frac{u}{A}\right)$$

where we can take B to be the constant

$$B := \sqrt{16m^{3/2}|V|^2 K_1 + 2mK_2 + 3mKK_2 + 14m^{3/2}nKK_1 + 100n^2m^3(K_1 + K_2)^2}.$$

Proof part (b) of Theorem 6.1. Consider the cutoff function ψ constructed in [18], which is supported in $Q_{R,T}(x_0, t_0)$, equals 1 in $Q_{R/2,T/2}(x_0, t_0)$, and satisfies

$$|\nabla\psi| \leq \frac{C}{R}, \qquad |\Delta_V\psi| \leq C\frac{1 + R\sqrt{K}}{R^2}, \qquad \frac{|\partial_t\psi|}{\sqrt{\psi}} \leq \frac{C}{T}, \qquad \frac{|\nabla\psi|^2}{\psi} \leq \frac{C}{R^2}$$

where C is a positive constant depending only on n. As in [18], we may require that $t_0 = T$ and ψ is supported in the slightly shorter space time cube $Q_{R,3T/4}(x_0, t_0)$.

For any smooth function η, as in the page 11 of [18], we have

$$\Box_{V,\psi}(\psi\eta) = \psi\Box_V\eta + \eta\Box_{V,\psi}\psi \tag{6.34}$$

where

$$\Box_{V,\psi} := \Box_V + \frac{2}{\psi}\langle\nabla\psi, \nabla\rangle. \tag{6.35}$$

Choosing $\eta = \lambda$ defined in (6.22) and using the evolution equation of λ, we have

$$\Box_{V,\psi}(\psi\lambda) = -\psi[H + (\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi})] + \lambda\Box_{V,\psi}\psi - \psi((\boldsymbol{V} \oplus_\alpha \boldsymbol{W})\boldsymbol{A})(\boldsymbol{\xi}, \boldsymbol{\xi}) + \psi(\boldsymbol{A}(\boldsymbol{V} \oplus_\alpha \boldsymbol{W}))(\boldsymbol{\xi}, \boldsymbol{\xi}), \tag{6.36}$$

where

$$H := \alpha(1-f)w\boldsymbol{V}(\boldsymbol{\xi}, \boldsymbol{\xi}) + 2(1-f)w\boldsymbol{W}(\boldsymbol{\xi}, \boldsymbol{\xi}) + 2|(\boldsymbol{V} + f\boldsymbol{W})\boldsymbol{\xi}|^2. \tag{6.37}$$

Given a positive constant β, consider a unit eigenvector $\boldsymbol{\xi}$ of $\psi(\boldsymbol{V} \oplus_\alpha \boldsymbol{W}) + \beta f\boldsymbol{g}$ with the maximal eigenvalue μ_m at the point (p_1, x_1). Extend $\boldsymbol{\xi}$ to be a vector field, still denoted by $\boldsymbol{\xi}$, by parallel translation along geodesics from p_1. Let $\mu_1, \ldots, \mu_m$ be the eigenvalues of the two form $\psi(\boldsymbol{V} \oplus_\alpha \boldsymbol{W}) + \beta f\boldsymbol{g}$ at (p_1, t_1) with the increasing order. As before, we may assume that $\mu_1 < 0 < \mu_m$. Define

$$\mu := [\psi\boldsymbol{V} \oplus_\alpha \boldsymbol{W} + \beta f\boldsymbol{g}](\boldsymbol{\xi}, \boldsymbol{\xi}) = \psi\lambda + \beta f. \tag{6.38}$$

Note that $\mu = \mu_m$ at the point (p_1, t_1). From (6.36) we get

$$\psi\Box_{V,\psi}\mu = -\psi^2[H^2 + (\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi})] + \psi\lambda\Box_{V,\psi}\psi + \psi\beta\Box_{V,\psi}f. \tag{6.39}$$

By definition, $\Box_{V,\psi}\psi$ is equal to

$$\begin{aligned}\Box_{V,\psi}\psi &= \partial_t\psi - \Delta_V\psi + \frac{2f}{1-f}\langle\nabla f, \nabla\psi\rangle + \frac{2}{\psi}|\nabla\psi|^2 \\ &= \partial_t\psi - \Delta_V\psi + \frac{2f}{1-f}\left\langle\sqrt{\psi}\nabla f, \frac{\nabla\psi}{\sqrt{\psi}}\right\rangle + \frac{2}{\psi}|\nabla\psi|^2.\end{aligned}$$

Without loss of generality, we may assume that $0 < u \leq A/e^3$; otherwise, for $\frac{A}{e^3} \leq u \leq A$ we can consider a new function $\tilde{u} := u/e^3 \in (0, A/e^3]$ and hence $\tilde{u}$ also satisfies the same estimate (6.2) which implies (6.2) for u. Under our hypothesis and (6.5), we arrive at

$$\Box_V f = \partial_t f - \Delta_V f + \frac{2f}{1-f}|\nabla f|^2 = |\nabla f|^2 + \frac{2f}{1-f}|\nabla f|^2 = \frac{1+f}{1-f}|\nabla f|^2 \leq -\frac{1}{2}|\nabla f|^2.$$

Consequently,

$$\psi\Box_{V,\psi}f = \psi\Box_V f + 2\left\langle\frac{\nabla\psi}{\sqrt{\psi}}, \sqrt{\psi}\nabla f\right\rangle \leq -\frac{1}{4}\psi|\nabla f|^2 + 4\frac{|\nabla\psi|^2}{\psi}.$$

As the estimate (6.23) (or see the page 13 in [18]) we have (since $\mu \geq 0$ implies $\psi\lambda \geq -\beta f \geq 0$)

$$-\psi^2 H \leq -\frac{2(\psi\lambda)^2}{\alpha^2} \quad \text{at } p_1, \text{ whenever } \mu \geq 0.$$

Hence, at the point (p_1, t_1),

$$\begin{aligned} 0 &\leq \psi \Box_{V,\psi}\mu \\ &\leq -\frac{2(\psi\lambda)^2}{\alpha^2} - \psi^2(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi}) + \beta\left(-\frac{1}{4}\psi|\nabla f|^2 + 4\frac{|\nabla\psi|^2}{\psi}\right) \\ &\quad + \left[|\partial_t \psi| + |\Delta_V \psi| + 2\frac{|\nabla\psi|^2}{\psi} + 2\sqrt{\psi}|\nabla f|\frac{|\nabla\psi|}{\sqrt{\psi}}\right]\psi\lambda \\ &\leq -\frac{(\psi\lambda)^2}{\alpha^2} - \psi^2(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi}) + \beta\left(-\frac{1}{4}\psi|\nabla f|^2 + 4\frac{|\nabla\psi|^2}{\psi}\right) \\ &\quad + \frac{1}{2}\left(|\partial_t\psi| + |\Delta_V\psi| + 2\frac{|\nabla\psi|^2}{\psi}\right)^2 + \frac{1}{2}\psi|\nabla f|^2\frac{|\nabla\psi|^2}{\psi}. \end{aligned}$$

Choosing

$$\beta := 2\sup_{\mathcal{M}} \frac{|\nabla\psi|^2}{\psi} \tag{6.40}$$

the above inequality shows that

$$0 \leq -\frac{(\psi\lambda)^2}{\alpha^2} - \psi^2(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi}) + \frac{1}{2}\left(|\partial_t\psi| + |\Delta_V\psi| + 2\frac{|\nabla\psi|^2}{\psi}\right)^2 + 8\sup_{\mathcal{M}}\frac{|\nabla\psi|^4}{\psi^2}$$

at the point (p_1, t_1). By the properties of the cutoff function ψ, we arrive at

$$\frac{(\psi\lambda)^2}{\alpha^2} \leq \psi^2(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi}) + 8C\left(\frac{1}{T} + \frac{1 + R\sqrt{K}}{R^2}\right)^2. \tag{6.41}$$

By the same calculation as that of (6.29), we obtain

$$\psi|(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi})| \leq 2m^{3/2}K_1(\mu_m + |\mu_1| + \beta|f|) + 4m(K_1 + K_2)\psi|\boldsymbol{W}| + 2m\psi K_2$$

at the point (p_1, t_1). Using $\mu_m + |\mu_1| \leq m\mu_m - \psi\frac{\alpha\Delta u}{u(1-f)} - \frac{\psi|\nabla u|^2}{u^2(1-f)^2} + m\beta|f|$, the above estimates imply

$$\psi|(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi})| \leq 4m(K_1 + K_2)\psi|\boldsymbol{W}| + 2m\psi K_2 + 2m^{3/2}K_1\left(m\mu_m - \psi\frac{\alpha\Delta u}{u(1-f)} - \frac{\psi|\nabla u|^2}{u^2(1-f)^2} + (m+1)\beta|f|\right)$$

at the point (p_1, t_1). Letting $a = q = 0$ in Theorem 5.3, for any $\alpha \geq 4$, we get

$$\psi\left(\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u}\right) \leq Cn^2\alpha^4\left(\frac{1}{T} + \frac{1 + R\sqrt{K}}{R^2} + K\right) \tag{6.42}$$

for some positive universal constant C, since the cutoff function is supported in a shorter cube. Using (6.42) we have

$$\begin{aligned} \psi^2|(\boldsymbol{P} \oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi}, \boldsymbol{\xi})| \leq\ & 2m^{5/2}K_1\psi^2\lambda + 2Cn^2m^{3/2}\alpha^4K_1\psi^2\left(\frac{1}{T} + \frac{1 + R\sqrt{K}}{R^2}\right) \\ & + \left[2mK_2 + \frac{m^{3/2}K_1\alpha^2}{2}|V|^2 + 2Cn^2m^{3/2}\alpha^4K_1K\right] \\ & + 4m(K_1 + K_2)\psi^2|\boldsymbol{W}| + 2m^{3/2}(m+1)K_1\beta|f|. \end{aligned}$$

According to Theorem 5.1 in [2] or [35], we can find a constant C' depending only on m so that

$$\psi^2|\boldsymbol{W}| \leq C'\left(\frac{1}{T} + \frac{1}{R^2} + K\right).$$

Consequently,

$$\psi^2|(\boldsymbol{P}\oplus_\alpha \boldsymbol{Q})(\boldsymbol{\xi},\boldsymbol{\xi})| \le 2m^{\frac{5}{2}}K_1\psi^2\lambda + C''m^{\frac{3}{2}}n^2\alpha^4(K_1+K_2)\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}\right)$$
$$+\left[2mK_2+\frac{m^{3/2}K_1\alpha^2}{2}|V|^2+2Cm^{3/2}n^2\alpha^4K_1K+4mC'(K_1+K_2)K\right]+4m^{5/2}K_1\beta|f|$$

for another positive universal constant C''. Plugging it into (6.41) implies

$$\frac{(\psi\lambda)^2}{\alpha^2} \le 2m^{5/2}K_1\psi\lambda + B_1\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}\right)+8C\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}\right)^2+B_2+4m^{5/2}K_1\beta|f|,$$

at the point (p_1,t_1), where

$$B_1 := C''m^{3/2}n^2\alpha^4(K_1+K_2),$$
$$B_2 := 2mK_2+\frac{m^{3/2}K_1\alpha^2}{2}|V|^2+2Cm^{3/2}n^2\alpha^4K_1K+4mC'(K_1+K_2)K.$$

An elementary inequality shows that

$$\frac{\psi\lambda}{\alpha} \le 2\alpha m^{5/2}K_1+\sqrt{8C}\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}+\frac{B_1}{16C}\right)+\sqrt{B_2}+2m^{5/4}\sqrt{K_1\beta|f|}$$

at the point (p_1,t_1). Therefore

$$\psi\lambda \le \sqrt{8C}\alpha\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}+B\right)+2m^{5/4}\sqrt{K_1\beta|f|}$$

at the point (p_1,t_1), where

$$B := \frac{2\alpha^2m^{5/2}K_1+\alpha\sqrt{B_2}}{\sqrt{8C}}+\frac{B_1\alpha}{16C}.$$

As the same argument in the page 16 of [18], using the inequality $2m^{5/4}\sqrt{K_1\beta|f|} \le \beta|f|+2m^{5/2}K_1$ and $f<0$, we must have

$$\mu \le \sqrt{8C}\alpha\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}+B+\frac{2m^{5/2}K_1}{\sqrt{8C}\alpha}\right) \quad \text{in } Q_{R,T}(x_0,t_0).$$

For any unit tangent vector $\boldsymbol{\xi}$ at x with $(x,t)\in Q_{R,T}(x_0,t_0)$, we have

$$\psi\boldsymbol{V}(\boldsymbol{\xi},\boldsymbol{\xi}) \le \sqrt{8C}\alpha\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}+B+\frac{2m^{5/2}K_1}{\sqrt{8C}\alpha}\right)(1-f) \quad \text{in } Q_{R,T}(x_0,t_0).$$

Taking $\alpha=4$ as in the proof of part (a), we finally obtain the following estimate

$$\psi\boldsymbol{V}(\boldsymbol{\xi},\boldsymbol{\xi}) \le C_1\left(\frac{1}{T}+\frac{1+R\sqrt{K}}{R^2}+B'\right)(1-f) \quad \text{in } Q_{R,T}(x_0,t_0)$$

where

$$B' := C_2m^{5/2}n^2\left[K_1+K_2+\sqrt{(K_1+K_2)K+K_2+K_1|V|^2}\right],$$

for some positive universal constants C_1, C_2.

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